6-CONNECTED GRAPHS ARE TWO-THREE LINKED

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No, emptiness is not nothingness. Emptiness is a type of existence. You must use this existential emptiness to fill yourself.

Liu Cixin, The Three-Body Problem

To my parents and my wife.

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SUMMARY

Let G be a graph and a_0, a_1, a_2, b_1, and b_2 be distinct vertices of G. Motivated by their work on Four Color Theorem, Hadwiger's conjecture for K_6, and Jørgensen's conjecture, Robertson and Seymour asked when does G contain disjoint connected subgraphs G_1, G_2, such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. We prove that if G is 6-connected then such G_1, G_2 exist. Joint work with Robin Thomas and Xingxing Yu.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Introduction to Hadwiger's conjecture and 2-3 linked graphs

The Four Color Theorem [1, 2, 3] asserts that every loopless planar graph admits a vertex 4-coloring. The related problem was first put forward by Francis Guthrie in 1852, who asked whether it is true that any planar map can be colored with four colors such that adjacent regions receive different colors. In 1976, Appel and Haken [1] claimed a proof of the Four Color Theorem with the help of a computer. However, some computer-free parts of their proof are complicated and tedious to verify. In 1997, Robertson, Sanders, Seymour, and Thomas [2, 3] gave a much simpler proof for the Four Color Theorem.

According to Kuratowski's theorem [4], a graph is planar if and only if it contains no K_5-subdivision or $K_{3,3}$-subdivision. Moreover, it is well known that any 3-connected nonplanar graph other than K_5 contains a $K_{3,3}$-subdivision. Hence, as an extension of the Four Color Theorem, it is natural to ask whether every graph without K_5-subdivision is also 4-colorable. More generally, Hajós [5] conjectured that for any positive integer k, any graph containing no K_{k+1}-subdivision is k-colorable. This conjecture is true for $k \leq 3$, but Catlin [5] found counterexamples to this conjecture for each $k \geq 6$. However, the cases for $k = 4$ and $k = 5$ are still open. Efforts have been made to resolve Hajós' conjecture for $k = 4$. Yu and Zickfeld [6] proved that a minimum counterexample to Hajós' conjecture when $k = 4$ must be 4-connected. Moreover, Sun and Yu [7] showed that if G is a minimum counterexample to Hajós' conjecture and S is a 4-cut in G then $G - S$ has exactly two components. In fact, if one can show a minimum counterexample to Hajós' conjecture for $k = 4$ is 5-connected, then Hajós' conjecture for $k = 4$ will immediately follow from the Kelmans-Seymour conjecture [8, 9]: Every 5-connected nonplanar graph contains K_5-subdivision. This Kelmans-Seymour conjecture was recently proved by He, Wang, and Yu [10, 11, 12, 13].

While Hajós' conjecture concerns the chromatic number of graphs with no K_{k+1}-subdivision, Hadwiger [14], in 1943, conjectured a far-reaching generalization of the Four Color Theorem in terms of K_{k+1}-minor: For any positive integer k, if a graph contains no K_{k+1}-minor then it is k-colorable.

It is easy to prove that Hadwiger's conjecture holds for $k \leq 2$. Hadwiger [14] and Dirac [15] proved the case for $k = 3$. For $k = 4$, Hadwiger's conjecture is equivalent to the Four Color Theorem by the result of Wagner [16], which characterized graphs containing no K_5-minor and showed that Four Color Theorem implies that graphs containing no K_5-minor are 4-colorable. The case $k = 5$ can also be reduced to the Four Color Theorem, as shown by Robertson, Seymour, and Thomas [17]. However, this conjecture remains open for $k \geq 6$.

In fact, there are also many other interesting results related to Hadwiger's conjecture. Suppose Hadwiger's conjecture is false for some k, and let G be a minor minimal counterexample. Dirac [15] showed that G is 5-connected when $k \geq 5$, and Mader [18] showed that G is 6-connected when $k \geq 5$, and 7-connected when $k \geq 6$. Kawarabayashi and G. Yu [19] proved that G is $(2k/27)$-connected, improving upon an earlier bound in [20].

Let the *stability number* $\alpha(G)$ of a graph G denote the size of the largest stable set in G. Then every n-vertex graph G has chromatic number at least $\lceil n/\alpha(G) \rceil$, and should contain a clique minor of this size if Hadwiger's conjecture is true. In 1982, Duchet and Meyniel [21] proved that every n-vertex graph G has a K_k-minor where $k \geq n/(2\alpha(G) - 1)$. Moreover, there has been a subsequent improvement by Fox [22]. And then Balogh and Kostochka [23] further improved the result, and showed that every n-vertex graph G has a K_k-minor where $k \geq 0.51338n/\alpha(G)$. Later, in 2007, Kawarabayashi and Song [24] proved that every n-vertex graph G with $\alpha(G) \geq 3$ has a K_k-minor where $k \geq n/(2\alpha(G) - 2)$.

For an n-vertex graph G with $\alpha(G) = 2$, the Duchet-Meyniel theorem implies that there is a K_k-minor with $k \geq n/3$, which was strengthened by Böhme, Kostochka and Thomason [25] in 2011. They proved that every n-vertex graph with chromatic number t has a K_k-minor where $k \geq (4t - n)/3$.

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. So graphs with stability number two are claw-free. Fradkin [26] showed that every n-vertex connected claw-free graph G with $\alpha(G) \geq 3$ has a K_k-minor where $k \geq n/\alpha(G)$. Furthermore, in 2010, Chudnovsky and Fradkin [27] proved that every claw-free graph G with no K_{k+1}-minor is $\lfloor 3k/2 \rfloor$-colorable.

Since line graphs are claw-free, these results about claw-free graphs are related to a theorem of Reed and Seymour. They showed [28] that Hadwiger's conjecture is true for line graphs (of multigraphs).

We say that H is an *odd minor* of G if H can be obtained from a subgraph G' of G by contracting a set of edges that is a cut of G'. Clearly, a graph contains K_3 as an odd minor if and only if it is not 2-colorable. In 1979, Catlin [5] showed that if G has no K_4 odd minor then G is 3-colorable. A *fully odd* K_4 in G is a subgraph of G which is obtained from K_4 by replacing each edge of K_4 by a path of odd length in such a way that the interiors of these six paths are disjoint. Zang [29] in 1998 and, independently, Thomassen [30] in 2001 proved the conjecture of Toft [31] that if G contains no fully odd K_4 then G is 3-colorable. In 1995, Gerards and Seymour conjectured a strenthening of Hadwiger's conjecture (see [32]) that for every $k \geq 0$, if G has no K_{k+1} odd minor, then G is k-colorable, which is known to be true for $k \leq 3$. More interesting results and open problems about Hadwiger's conjecture and its variations can be found in [33], which was written by Seymour in 2016.

Now, we come back and spend a bit more space on the $k = 5$ case of the Hadwiger conjecture. As we mentioned, Mader [18] proved that any minor minimal counterexample to the Hadwiger conjecture for $k = 5$ is 6-connected. Jørgensen [34] conjectured that every 6-connected graph contains a K_6-minor or has a vertex whose removal results in a planar graph. Therefore, if Jørgensen's conjecture holds, then Hadwiger's conjecture for $k = 5$ easily reduces to the Four Color Theorem. In 2017, Kawarabayashi, Norine, Thomas, and Wollan [35] showed that Jørgensen's conjecture holds for sufficiently large graphs.

In their work [17], Robertson, Seymour, and Thomas proved that Jørgensen's conjecture holds for each 6-connected graph in which some edge is contained in four triangles. (However, they were not able to resolve the Jørgensen conjecture. Instead, they explored different structures of a minimum counterexample to the Hadwiger conjecture.) It is natural and useful to extend this result to graphs in which some edge is contained in three triangles: Given a 6-connected graph G and triangles $a_ib_1b_2a_i$ for $i = 0, 1, 2$ in G, can we prove that G contains K_6-minor or has a vertex whose removal results in a planar graph?

A first step is to prove that 6-connected graphs are *two-three linked*: If G is a 6-connected graph and a_0, a_1, a_2, b_1, b_2 are distinct vertices of G, then G contains disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. We believe that we have such a characterization except that it is quite complicated (even to state) and its proof is long.

1.2 A main theorem about 2-3 linked graphs

For convenience, we use $(G, a_0, a_1, a_2, b_1, b_2)$ to denote a graph G and distinct vertices a_0, a_1, a_2, b_1, b_2 of G, and call it a *rooted graph*. A *cluster* in a graph G is a set $\mathcal{X}$ of disjoint subsets of $V(G)$ such that each member of $\mathcal{X}$ induces a connected subgraph of G. We say that a rooted graph $(G, a_0, a_1, a_2, b_1, b_2)$ is *feasible* if there exists a cluster $\{X_1, X_2\}$ in G such that $\{a_0, a_1, a_2\} \subseteq X_1$ and $\{b_1, b_2\} \subseteq X_2$. We can now state our result as follows.

Theorem 1.2.1 *Let* $(G, a_0, a_1, a_2, b_1, b_2)$ *be a rooted graph, and assume* $G + b_1b_2 + \{a_ib_j : i = 0, 1, 2$ *and* $j = 1, 2\}$ *is 6-connected. Then* $(G, a_0, a_1, a_2, b_1, b_2)$ *is feasible.*

We may view the problem of characterizing feasible rooted graphs as a generalization of the following problem of characterizing 2-linked graphs: Given a graph G and four distinct vertices a_1, a_2, b_1, b_2 of G, when does G contain disjoint paths from a_1, a_2 to b_1, b_2, respectively? Several characterizations of 2-linked graphs are known in [36, 37, 38, 39] and have been used extensively in the literature for proving important structural results on graphs (e.g., in the graph minors project of Robertson and Seymour).

Suppose $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$ is an infeasible rooted graph such that $b_1b_2 \notin E(G)$, $a_ib_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected. A *B-bridge* of G is a subgraph of G induced by all edges in a component of $G - V(B)$ and all edges from that component to B.

In Chapter 2, we will present the proof of our main theorem, and in Chapter 3, some future works will be introduced.

In fact, in section 2.1, we show that for some $i \in \{0, 1, 2\}$, G has an *a_i-frame* A, B in $(G, a_0, a_1, a_2, b_1, b_2)$, that is $G - a_i$ has disjoint paths A from a_{i-1} to a_{i+1} and B from b_1 to b_2 (with $a_{-1} = a_2, a_3 = a_0$). Moreover, given an a_i-frame A, B for some $i \in \{0, 1, 2\}$, we will prove some useful properties. For example, we prove that the B-bridge of G containing a_i can be drawn in a disk in which no two edges cross, and b_1, b_2, a_i occur on the boundary of the disk.

In section 2.2, we further show that γ has a *good frame* and an *ideal frame*. For an ideal a_i-frame A, B in γ, roughly speaking, we group the $(A \cup B)$-bridges of G not containing a_i into *slim connectors* and *fat connectors*.

In sections 2.3 and 2.4, we deal with the case when there exists at least one fat connector in A, B. In section 2.5, we solve the case when there does not exist any fat connector. In this case, $G - A$ can be drawn in a disk in which no two edges cross, b_1, b_2, a_i occur on the boundary of the disk, and any A-B path in G is induced by a single edge. So the structure of G is quite simple in some sense. However, in both cases, we will try to find a configuration consisting of paths with special properties, and use them to force a small cut in G or show that $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible.

For readers' convenience, we also draw Figure 1.1 containing the illustration of structures of some important special graphs, which shows a sketch of our proof idea.

Finally, we end this chapter with some notation and terminology. Let G_1, G_2 be two graphs. We use $G_1 \cup G_2$ (respectively, $G_1 \cap G_2$) to denote the graph with vertex set $V(G_1) \cup V(G_2)$ (respectively, $V(G_1) \cap V(G_2)$) and edge set $E(G_1) \cup E(G_2)$ (respectively, $E(G_1) \cap E(G_2)$). Let G be a graph, a *separation* in G is a pair (G_1, G_2) of edge-disjoint subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$. And $|V(G_1) \cap V(G_2)|$ is the *order* of the separation (G_1, G_2). Let P be a path, and let $u, v \in V(P)$. Then we write $P[u, v) := P[u, v] - v$, $P(u, v] := P[u, v] - u$, and $P(u, v) := P[u, v] - \{u, v\}$. For any positive integer m, we let $[m] := \{1, \cdots, m\}$.

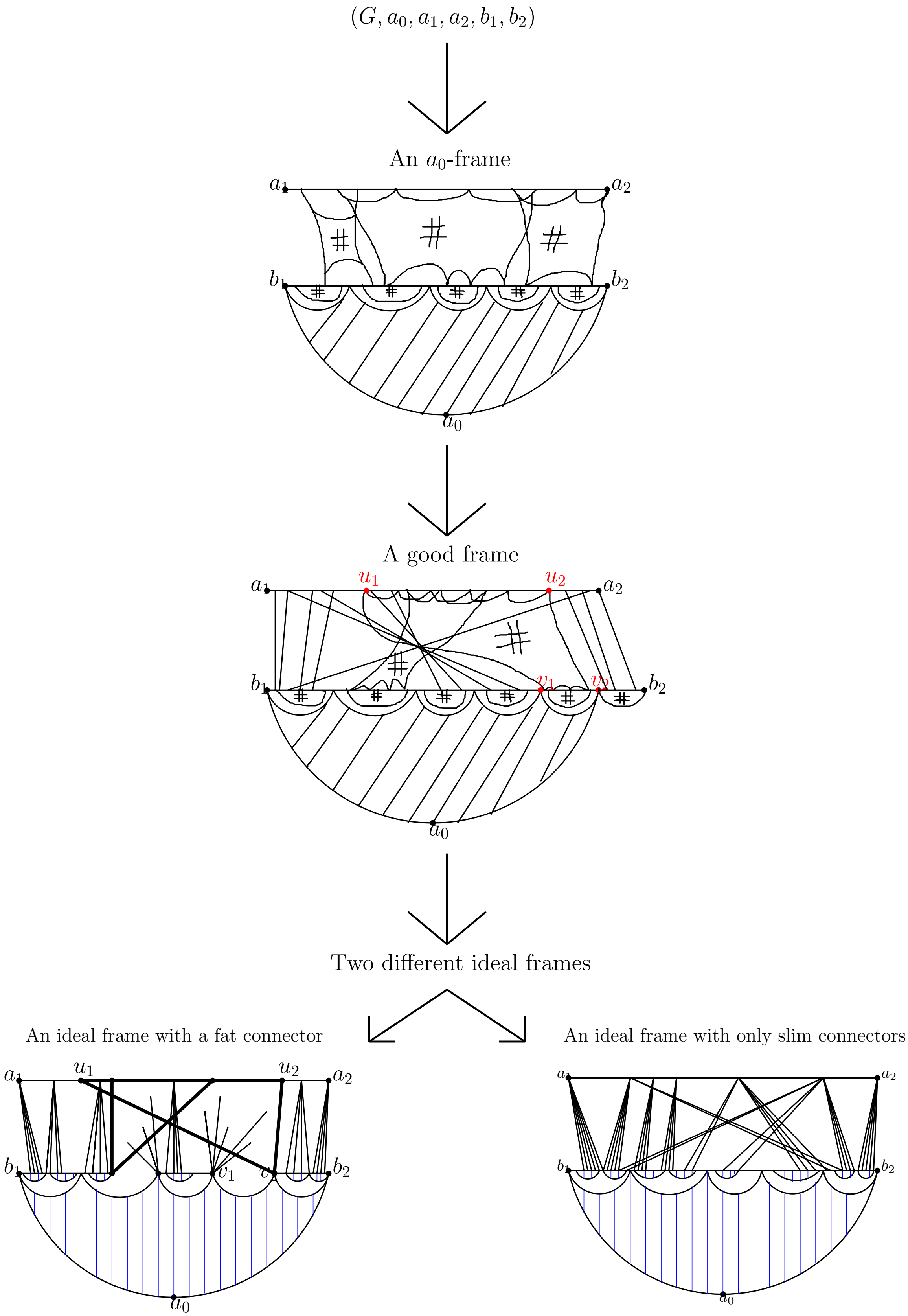

Figure 1.1: A flow chart of proof

CHAPTER 2

THE PROOF OF MAIN THEOREM

2.1 Frames

In the first section of this chapter, we state some known results and prove some lemmas that we will use. In particular, we show that an infeasible rooted graph must contain a "frame" which consists of two disjoint paths.

A result we use often is Seymour's characterization of 2-linked graphs [37] (with equivalent versions in [36, 38, 39]). To state this result we introduce several concepts. A *disk representation* of a graph G is a drawing of G in a disk in which no two edges cross. A 3-*planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, ..., A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \varnothing$) such that

(i) for $i \neq j$, $N_G(A_i) \cap A_j = \varnothing$,

(ii) for $1 \leq i \leq k$, $|N_G(A_i)| \leq 3$, and

(iii) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding edges joining every pair of distinct vertices in $N_G(A_i)$, then $p(G, \mathcal{A})$ can be drawn in the plane without crossing edges.

If, in addition, $b_0, b_1, ..., b_n$ are vertices in G such that $b_i \notin A$ for $0 \leq i \leq n$ and $A \in \mathcal{A}$, $p(G, \mathcal{A})$ can be drawn in a closed disk with no edge crossings, and $b_0, b_1, ..., b_n$ occur on the boundary of the disk in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, ..., b_n)$ is 3 -planar. If there is no need to specify $\mathcal{A}$, we may simply say that $(G, b_0, b_1, ..., b_n)$ is 3-planar. If $\mathcal{A} = \emptyset$, we say that $(G, b_0, b_1, ..., b_n)$ is planar. Moreover, we say that a face of (the disk representation of) G is *finite*, if the face is inside the disk.

Lemma 2.1.1 (Seymour, 1980) *Let G be a graph with distinct vertices x_1, x_2, x_3, x_4. Then either (G, x_1, x_2, x_3, x_4) is 3-planar, or G has a cluster $\{X_1, X_2\}$ such that $\{x_1, x_3\} \subseteq X_1$ and $\{x_2, x_4\} \subseteq X_2$.*

We say that a sequence $(\alpha_1, \cdots, \alpha_n)$ is larger than $(\beta_1, \cdots, \beta_m)$ with respect to the lexicographic ordering if either

(i) $m < n$ and $\alpha_i = \beta_i$ for $i = 1, \cdots, m$, or

(ii) there exists $j \in [min(m, n)]$ with $\alpha_j > \beta_j$ and $\alpha_i = \beta_i$ for all $i < j$.

We will also use the following lemma to modify paths.

Lemma 2.1.2 *Let G be a connected graph and P be a path between vertices u_1 and u_2 of G, and let C denote a component of $G - P$. Then one of the following holds:*

- *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $V(C) \cup \{u_1, u_2\} \subseteq V(G_1)$, and $|V(G_2 - G_1)| \geq 1$, or*

- *G has an induced path Q from u_1 to u_2 such that $G - Q$ is connected with $C \subseteq (G - Q)$.*

Proof. We choose a path Q in G from u_1 to u_2 and label the components of $G - Q$ as $C_1, \ldots, C_n$ such that $C \subseteq C_1$ and $|V(C_2)| \geq \cdots \geq |V(C_n)|$, and, subject to this, $s(Q) := (|V(C_1)|, |V(C_2)|, \cdots, |V(C_n)|)$ is maximum under the lexicographical ordering. Note that Q is well defined because of P.

Then Q is an induced path in G. For, otherwise, let Q' be the induced path in $G[Q]$ from u_1 to u_2 then $s(Q') > s(Q)$, a contradiction. If $n = 1$ then the assertion of the lemma holds. So assume $n \geq 2$.

Let $l_n, r_n \in N_G(C_n) \cap V(Q)$ such that $Q[l_n, r_n]$ is maximal. We may assume there exists C_j with $j < n$ such that $N_G(C_j) \cap Q(l_n, r_n) \neq \emptyset$; otherwise, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{l_n, r_n\}$, $V(C) \cup \{u_1, u_2\} \subseteq V(G_1)$, and $V(C_n) \subseteq V(G_2 - G_1)$, a contradiction.

Now let Q' be an induced path between u_1 and u_2 in $G[Q \cup C_n]$ such that $Q' \cap Q(l_n, r_n) = \emptyset$. Clearly, $s(Q') > s(Q)$ under the lexicographical ordering, a contradiction. $\square$

In the remainder of this paper, we will always assume that

- $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$ is a given rooted graph such that $b_1 b_2 \notin E(G)$, $a_i b_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and
- $G^* := G + b_1 b_2 + \{a_i b_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected.

When we write a_{i+j}, we understand that the subscript $i + j$ is taken modulo 3. In the next two lemmas, we show that G does not admit certain separations.

Lemma 2.1.3 *G has no separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $|V(G_2 - G_1)| \geq 2$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, c_1, c_2, c_3, c_4, c_5, c_6)$ is planar.*

Proof. For, otherwise, let $G_2' := G_2 + \{c_1c_2, c_2c_3, c_3c_4, c_4c_5, c_5c_6, c_6c_1, c_1c_3, c_3c_5, c_5c_1\}$, which is planar as $(G_2, c_1, c_2, c_3, c_4, c_5, c_6)$ is planar.

Since G^* is 6-connected, G_2 has at least one edge from each c_i to $V(G_2 - G_1)$ and, hence, the number of edges in G_2 with at least one end in $V(G_2 - G_1)$ is at least $(6|V(G_2 - G_1)| + 6)/2 = 3|V(G_2 - G_1)| + 3 = 3|V(G_2)| - 15$. Thus, G_2' has at least $3|V(G_2)| - 15 + 9 = 3|V(G_2)| - 6$ edges.

Thus, G_2' is a planar graph with exactly $3|V(G_2')| - 6$ edges and each c_i has a unique neighbor in $G_2 - G_1$. Note that G_2' must be a planar triangulation. Therefore, the neighbors of $c_1, \cdots, c_6$ in $G_2 - G_1$ are the same. Hence, since G^* is 6-connected, $|V(G_2 - G_1)| = 1$, a contradiction. $\square$

Lemma 2.1.4 *G has no separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 4$ and for some permutation π of $\{0, 1, 2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $|V(G_2 - G_1)| \geq 4$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, V(G_1 \cap G_2))$ is planar.*

Proof. Suppose to the contrary that such a separation (G_1, G_2) exists in G and let $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$ such that $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1))$ is planar. Let $X :=$

$V(G_2 - G_1) - \{a_{\pi(0)}, a_{\pi(1)}, b_j\}$. Since G^* is 6-connected, we see that G_2 has at least two edges from b_j to X and at least three edges from $a_{\pi(i)}$ to X for $i \in \{0, 1\}$.

Further, for any $i \in [4]$, c_i has a neighbor in X. For, otherwise, suppose, for some $i \in [4]$, c_i has no neighbor in X. Then by applying Lemma 2.1.3 to the separation $(G[V(G_1) \cup \{c_i\}], G_2 - c_i)$ in G, we see that $|X| = 1$. It then follows from planarity that b_j has at most one neighbor in X, a contradiction.

Hence, the number of edges in G_2 with at least one end in X is at least $(6|X|+1+1+1+1+3+3+2)/2 = 3|X|+6$. So $G_2' := G_2+\{c_1c_2, c_2c_3, c_3c_4, c_4a_{\pi(1)}, a_{\pi(1)}b_j, b_ja_{\pi(0)}, a_{\pi(0)}c_1, c_2a_{\pi(0)}, c_2b_j, c_2c_4, c_4b_j\}$ has edges at least $3|X| + 6 + 11 = 3(|X| + 7) - 4$. On the other hand, since G_2' is planar (as $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)$ is planar), G_2' has at most $3(|X| + 7) - 6$ edges, a contradiction. $\square$

For $i \in \{0, 1, 2\}$, an *a_i-frame* in γ consists of disjoint paths A from a_{i-1} to a_{i+1} and B from b_1 to b_2 in $G - a_i$, such that A is induced in G, $G - A$ is connected, and the B-bridge of G containing a_i does not contain A. The next lemma says that if γ is infeasible then it has a frame.

Lemma 2.1.5 *If γ is infeasible then there exists $i \in \{0, 1, 2\}$ such that γ has an a_i-frame.*

Proof. Since G^* is 6-connected, $G - \{a_0, a_1, a_2\}$ contains an induced path P from b_1 to b_2 such that $G - \{a_0, a_1, a_2\} - P \neq \emptyset$. By Lemma 2.1.2, $G - \{a_0, a_1, a_2\}$ has an induced path Q from b_1 to b_2 such that $C := G - \{a_0, a_1, a_2\} - Q$ is connected and $C \neq \emptyset$.

Note that there exists a permutation i, j, k of $\{0, 1, 2\}$ such that $N_G(a_j) \cap V(C) \neq \emptyset$ and $N_G(a_k) \cap V(C) \neq \emptyset$, or $N_G(a_j) \cap V(C) = \emptyset$ and $N_G(a_k) \cap V(C) = \emptyset$. In the former case, $G - a_i$ contains disjoint paths from b_1, a_j to b_2, a_k, respectively. In the latter case, $N_G(a_j) \cap V(Q(b_1, b_2)) \neq \emptyset$ and $N_G(a_k) \cap V(Q(b_1, b_2)) \neq \emptyset$; so we have a path in $G[Q(b_1, b_2)+\{a_j, a_k\}]$ from a_j to a_k and a path from b_1 to b_2 in $G-\{a_0, a_1, a_2\}-Q(b_1, b_2)$.

Hence, there exists $i \in \{0, 1, 2\}$ such that $G - a_i$ has disjoint paths A^* and B from a_{i-1}, b_1 to a_{i+1}, b_2, respectively. Since γ is infeasible, a_i and A^* are contained in different components of $G - B$. Hence, a_i and B are contained in a component of $G - A^*$. So by Lemma 2.1.2, G has an induced path A between a_{i-1} and a_{i+1} such that $G - A$ is connected and $B + a_i \subseteq G - A$. Since γ is infeasible, the B-bridge of G containing a_i does not contain A. Hence, A, B is an a_i-frame in γ. $\square$

In the next two lemmas, we derive useful information about frames in γ, seen at Figure 2.1.

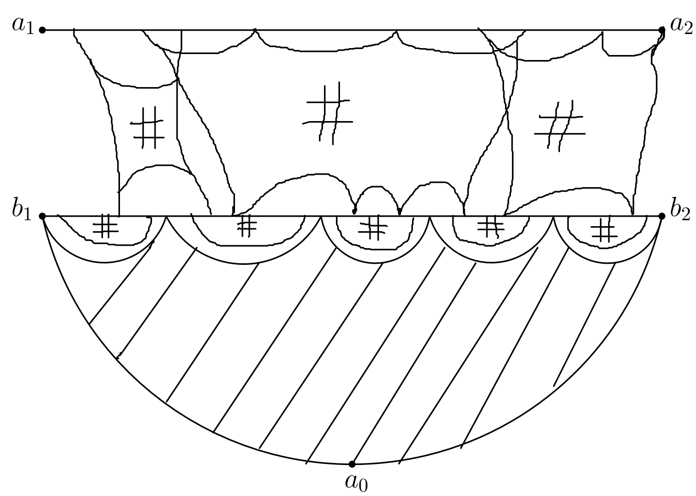

Figure 2.1: An a_0-frame

Lemma 2.1.6 *Suppose γ is infeasible and A, B is an a_i-frame in γ. Let $A_i(B)$ denote the B-bridge of G containing a_i, and let $V(A_i(B) \cap B) = \{d_1, \cdots, d_t\}$ such that $b_1, d_1, \cdots, d_t$, b_2 occur on B in this order. Then $(A_i(B) \cup B, a_i, b_1, d_1, \cdots, d_t, b_2)$ is planar.*

Proof. Let $G' = G/A$, and let a' denote the vertex representing the contraction of A. Since γ is infeasible, G' has no disjoint paths from a', b_1 to a_i, b_2, respectively. So by Lemma 2.1.1, there exists a set $\mathcal{S}$ of pairwise disjoint subsets of $V(G')$, such that $(G', \mathcal{S}, a'$, $b_1, a_i, b_2)$ is 3-planar.

Note that for any $S \in \mathcal{S}$, $a' \in N_{G'}(S)$. For, otherwise, $N_G(S)$ is a cut in G^* separating S from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction as G^* is 6-connected.

Thus, for any $S \in \mathcal{S}$, we have $|N_{G'}(S) \cap V(B)| \leq 2$. Hence, $S \cap A_i(B) = \emptyset$. For otherwise, since $a' \in N_{G'}(S)$, there exists $u \in V(A_i(B) \cap B)$, such that $u \in S$. But then $G - A$ contains three internally disjoint paths from u to b_1, b_2, a_i, respectively, a contradiction to the existence of cut $N_{G'}(S)$. Therefore, $A_i(B) \subseteq G' - \cup_{S \in \mathcal{S}} S$, and $G' - \cup_{S \in \mathcal{S}} S$ has a disk representation with b_1, b_2, a_i on the boundary of the disk. Thus, $A_i(B) \cup B$ inherits a disk representation with b_1, b_2, a_i occurring on the boundary of the disk. Since $A_i(B) \cup B - B$ has only one component, $(A_i(B) \cup B, a_i, b_1, d_1, \cdots, d_t, b_2)$ is planar. $\square$

Suppose A, B is an a_i-frame in γ. Let $A_i(B)$ denote the B-bridge of G containing a_i. By a *double cross* in A, B we mean a pair of disjoint connected subgraphs A', B' (in this order) of $G - (A_i(B) - B)$ for which there exist $a_1', a_2' \in V(A)$ and $b_1', b_2' \in V(B)$, such that $V(A')$ includes a_1', a_2' and at least one vertex of $B(b_1', b_2')$ and is otherwise disjoint from $A \cup B[b_1, b_1'] \cup B[b_2', b_2]$, and $V(B')$ includes b_1', b_2' and at least one vertex of $A(a_1', a_2')$ and is otherwise disjoint from $B \cup A[a_1, a_1'] \cup A[a_2', a_2]$. The vertices a_1', a_2', b_2', b_1' (in this order) are called the *terminals* of the double cross.

Lemma 2.1.7 *If γ is infeasible then there is no double cross in γ.*

Proof. Without loss of generality, assume A, B is an a_0-frame in γ. Suppose A', B' is a double cross in A, B with terminals a_1', a_2', b_2', b_1'. Let $H = A(a_1', a_2') \cup B(b_1', b_2') \cup (A' - \{a_1', a_2'\}) \cup (B' - \{b_1', b_2'\})$. Consider the graph G' obtained from G by contracting H to a single vertex h.

Since G^* is 6-connected, then, combined with the existence of four disjoint paths $A[a_1, a_1'], A[a_2', a_2], B[b_1, b_1'], B[b_2', b_2]$ and Menger's theorem, G' contains five vertex disjoint paths between $\{a_1', a_2', b_1', b_2', h\}$ and $\{a_0, a_1, a_2, b_1, b_2\}$. So G contains five disjoint paths P_i, $i = 1, \ldots, 5$, (also internally disjoint from H) joining a_1', a_2', b_1', b_2' and H to $\{a_0, a_1, a_2, b_1, b_2\}$. Without loss of generality, assume that $a_1 \in V(P_1)$, $a_2 \in V(P_2)$, $b_1 \in V(P_3)$, $b_2 \in V(P_4)$, and $a_0 \in V(P_5)$.

Let $S_1 = (V(P_1 \cup P_2 \cup P_5)) \cap (\{a_1', a_2', b_1', b_2'\} \cup V(H))$, and $S_2 = (V(P_3 \cup P_4)) \cap (\{a_1', a_2', b_1', b_2'\} \cup V(H))$. Using the properties of a double cross, we can show that H contains a cluster $\{H_1, H_2\}$ such that $S_i \subseteq V(H_i)$, $i = 1, 2$. Let $X_1 := H_1 \cup V(P_1 \cup P_2 \cup P_5)$ and $X_2 := V(P_3 \cup P_4) \cup H_2$. Then $\{X_1, X_2\}$ is a cluster in G, a contradiction. □

We conclude this section by considering intersections of special cuts in a planar graph, and investigating when they force another cut or interesting structures of the graph.

Lemma 2.1.8 *Let γ be infeasible with an a_0-frame A, B, and let G_0 be obtained from G^* by deleting the component of $G^* - B$ containing A. Suppose (G_0, a_0, b_1, B, b_2) is planar, and G_0 has 3-cuts $\{a_0', b_1', b_2'\}$ and $\{a_0'', b_1'', b_2''\}$ separating $\{a_0, b_1, b_2\}$ from $B[b_1', b_2']$ and $B[b_1'', b_2'']$, respectively, such that $b_1, b_1'', b_1', b_2'', b_2', b_2$ occur on B in order, $b_1' \neq b_2''$, and G_0 contains a path from $B(b_1', b_2'')$ to a_0 and internally disjoint from B. Then one of the following holds:*

(i) $\{b_1'', b_2'\}$ *is contained in a 3-cut of G_0 separating $\{a_0, b_1, b_2\}$ from $B[b_1'', b_2']$.*

(ii) $\{b_1'', b_2'\} = \{b_1, b_2\}$*, and $a_0' = a_0'' = a_0$.*

(iii) $\{a_0'', b_1'', b_2'\} = \{a_0, b_1, b_2\}$*, b_2'' is a cut vertex of G_0 separating b_2 from $\{a_0, b_1\}$, and a_0', a_0'', b_2', b_2'' are incident with some finite face of G_0.*

(iv) $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$*, b_1' is a cut vertex of G_0 separating b_1 from $\{a_0, b_2\}$, and a_0', a_0'', b_1', b_1'' are incident with some finite face of G_0.*

Proof. We may assume $a_0' \neq a_0''$. For, otherwsie, since (G_0, a_0, b_1, B, b_2) is planar, either $\{a_0', b_1'', b_2'\}$ is a 3-cut in G_0 separating $\{a_0, b_1, b_2\}$ from $B[b_1'', b_2']$ and (i) holds, or $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$ and (ii) holds.

For $i \in [2]$, let F_i' be a finite face of G_0 incident with both b_i' and a_0' and let F_i'' be a finite face of G_0 incident with both b_i'' and a_0''. Since $a_0' \neq a_0''$, $b_1, b_1'', b_1', b_2'', b_2'$ occur on B in order, and G_0 contains a path from $B(b_1'', b_2')$ to a_0 and internally disjoint from B, we have $F_i' = F_i''$ for some $i \in [2]$.

By symmetry, we may assume $F_1' = F_1''$. Then a_0', a_0'', b_1', b_1'' are incident with some finite face of G_0. Thus, either $\{a_0', b_1'', b_2'\}$ is a 3-cut of G_0 separating $\{a_0, b_1, b_2\}$ from $B[b_1'', b_2']$, or $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$ and b_1' is a cut vertex of G_0 separating b_1 from $\{a_0, b_2\}$. So (i) or (iv) holds, a contradiction. $\square$

Lemma 2.1.9 *Let γ be infeasible and A, B be an a_0-frame in γ, and let G_0 be obtained from G^* by deleting the component of $G^* - B$ containing A. Suppose (G_0, a_0, b_1, B, b_2) is planar, and G_0 has four distinct vertices b_1'', b_1', b_2'', b_2' with $b_1, b_1'', b_1', b_2'', b_2', b_2$ on B in order, and b_1'', b_2'' are incident with some finite face of G_0.*

(i) If $\{b_1', b_2'\}$ is a 2-cut in G_0 separating $B[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, then b_1'', b_1', b_2'', b_2' are incident with some finite face of G_0, and $\{b_1'', b_2'\}$ is a 2-cut in G_0 separating $B[b_1'', b_2']$ from $\{a_0, b_1, b_2\}$.

(ii) If there exists a vertex a_0' in G_0, such that $\{a_0', b_1', b_2'\}$ is a 3-cut in G_0 separating $B[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, then one of the following occurs:

(a) a_0', b_1'', b_1', b_2'' are incident with some finite face of G_0, and $\{a_0', b_1'', b_2'\}$ is a 3-cut in G_0 separating $B[b_1'', b_2']$ from $\{a_0, b_1, b_2\}$ or $\{a_0', b_1'', b_2'\} = \{a_0, b_1, b_2\}$;

(b) a_0', b_1'', b_2'', b_2' are incident with some finite face of G_0, and $\{b_1'', b_2'\}$ is a 2-cut in G_0 separating $B[b_1'', b_2']$ from $\{a_0, b_1, b_2\}$.

Proof. Let F'' be a finite face of G_0 incident with b_1'', b_2''. To prove (i), we let F' be a finite face of G_0 incident with b_1', b_2'. Since $b_1, b_1'', b_1', b_2'', b_2', b_2$ occur on B in order, $F' = F''$, and so (i) holds.

Next, we prove (ii). For each $i \in [2]$, we let F_i' be a finite face of G_0 incident with both b_i' and a_0'. Since $b_1, b_1'', b_1', b_2'', b_2', b_2$ occur on B in order, then $F_1' = F''$ or $F_2' = F''$. Now, if $F_1' = F''$, then (a) of (ii) holds; if $F_2' = F''$, then (b) of (ii) holds. $\square$

2.2 Good frames and ideal frames

In this section, we fix $\gamma = (G, a_0, a_1, a_2, b_1, b_2)$ and $G^* = G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$, assume that γ is infeasible, and then show that γ has a special frame with good properties. For an a_i-frame A, B in γ, we fix the following notation:

- $\alpha(A, B) = |\{b_i : N_G(b_i) \cap V(A_i(B) - a_i - B) \neq \emptyset\}|$, and
- $c(A, B) = |\{v \in V(A_i(B) \cap B) - \{b_1, b_2\} : \{v, a_i\}$ separates b_1 from b_2 in $A_i(B) \cup B\}|$.

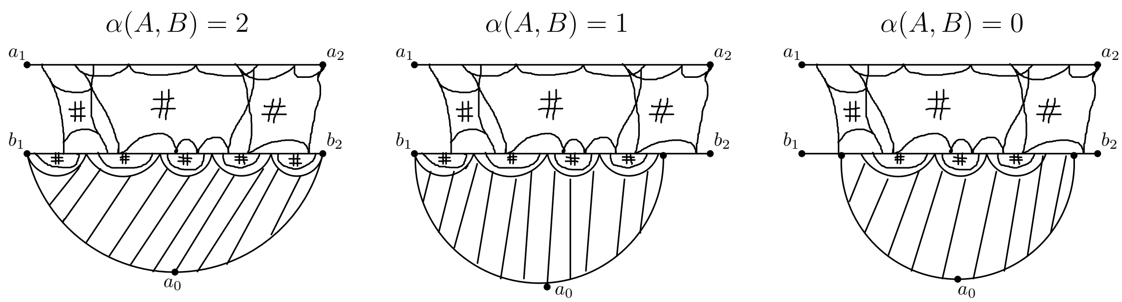

Figure 2.2: $\alpha(A, B)$

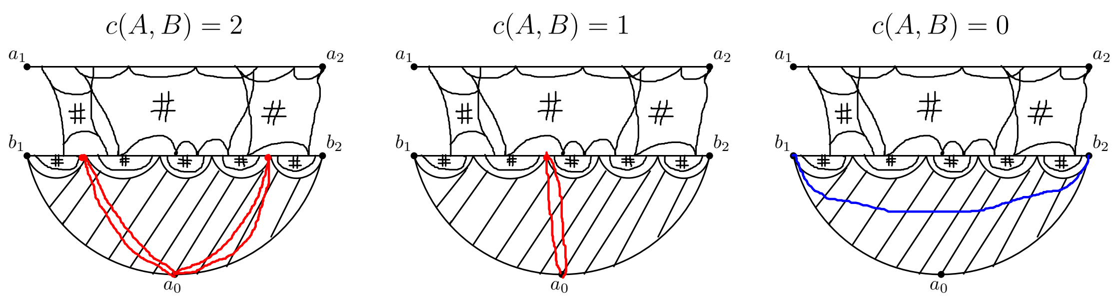

Figure 2.3: $c(A, B)$

We say that an a_i-frame A, B in γ is *good* (seen at Figure 2.4), if among all the frames in γ,

(i) $\alpha(A, B)$ is maximum,

(ii) subject to (i), $c(A, B)$ is minimum,

(iii) subject to (ii), $A_i(B)$ is maximal.

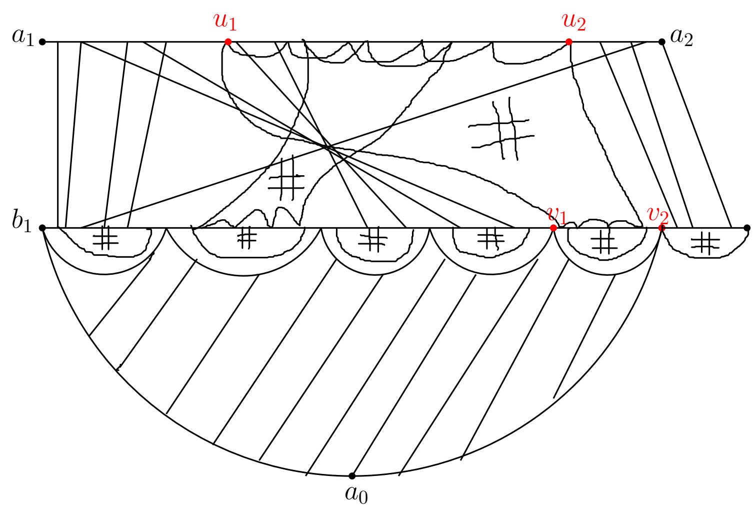

Figure 2.4: A good frame and its connectors

Lemma 2.2.1 *Suppose A, B is a good frame in γ. Let $i \in \{0, 1, 2\}$ and A', B' be disjoint paths in $G - a_i$ from a_{i-1}, b_1 to a_{i+1}, b_2, respectively.*

(i) If, for some $j \in [2]$, G has a path B_0 from a_i to b_j that is internally disjoint from A', B', then $\alpha(A, B) \geq 1$.

(ii) If $\{a_i, b_1, b_2\}$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, then $\alpha(A, B) = 2$.

(iii) If G has a path B'' from b_1 to b_2 that is internally disjoint from A', B', then $\alpha(A, B) = 2$ and $c(A, B) = 0$.

Proof. We first prove (i). We see that B', B_0 are contained in some component of $G - A'$. By Lemma 2.1.2 and the existence of A', there exists an induced path A^* from a_{i-1} to a_{i+1}, such that $G - A^*$ is connected, and $B', B_0 \subseteq G - A^*$. Since γ is infeasible, A^* and a_i

are in different components of $G - B'$. So A^*, B' is a frame. By the existence of B_0, $\alpha(A^*, B') \geq 1$, and so $\alpha(A, B) \geq 1$.

Similarly, for (ii), let C be the component of $G - (A' \cup (B' - \{b_1, b_2\}))$ containing b_1, b_2, a_i, we may assume there exists an induced path A^* from a_{i-1} to a_{i+1}, such that $G - A^*$ is connected, and $B', C \subseteq G - A^*$. So A^*, B' is a frame. By the existence of C, $\alpha(A^*, B') = 2$, and so $\alpha(A, B) = 2$.

For (iii), since γ is infeasible, $B' \cup B'' + a_i$ must be contained in a component of $G - A'$. Hence, we may assume that $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$. So by (ii), $\alpha(A, B) = 2$. Now by Lemma 2.1.2 and the existence of A', there exists an induced path A^* from a_{i-1} to a_{i+1}, such that $G - A^*$ is connected, and $B' \cup B'' + a_i \subseteq G - A^*$. So A^*, B' is a frame. Since $B'' + a_i$ is contained in a component of $G - (A' \cup (B' - \{b_1, b_2\}))$, we see that $c(A, B) = 0$. $\square$

For a frame A, B in γ, an A-B *bridge* is an $(A \cup B)$-bridge of G that intersects both A and B. Let M be an A-B bridge, $l, r \in V(A \cap M)$, and $l', r' \in V(B \cap M)$, such that $A[l, r]$ and $B[l', r']$ are maximal. Then we say that l, r are the *extreme hands* of M, and that l', r' are the *feet* of M. We say that M *lies on* $B[b'_1, b'_2]$ for some $b'_1, b'_2 \in V(B)$, if $B[l', r'] \subseteq B[b'_1, b'_2]$. We say that M is *fat* if $|V(M \cap B)| \geq 2$ and *non-fat* if $|V(M \cap B)| = 1$.

Lemma 2.2.2 *Suppose A, B is a good a_0-frame in γ. Let $\{d_1, \cdots, d_t\} = V(B \cap A_0(B))$ such that $b_1, d_1, \cdots, d_t, b_2$ occur on B in order, and let $d_0 = b_1, d_{t+1} = b_2$. Then the following conclusions hold:*

(i) For any $i \in [t]$, $G - (A_0(B) - (B - d_i)))$ does not contain disjoint paths from a_1, b_1 to a_2, b_2, respectively.

(ii) For any A-B bridge M, $M \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t+1]$.

(iii) Let N be a B-bridge of G not containing A or a_0, then $|V(N \cap B)| \geq 4$, and $N \cap B \subseteq B[d_{i-1}, d_i]$ for some $i \in [t+1]$.

Proof. First, we note that (ii) and (iii) follow immediately from (i). So we prove (i). Suppose (i) fails, and let A^*, B' be disjoint paths in $G - (A_0(B) - (B - d_i)))$ from a_1, b_1 to a_2, b_2, respectively.

Then $A_0(B) \cup B'$ is contained in a component of $G - A^*$. By Lemma 2.1.2 and the existence of A^*, there exists an induced path A' from a_1 to a_2, such that $G - A'$ is connected, and $A_0(B) \cup B' \subseteq G - A'$. So A', B' is a frame in γ. Now, due to the existence of d_i, the B-bridge of G containing a_0 is properly contained in the B'-bridge of G containing a_0, a contradiction. □

An a_i-frame A, B in γ is *ideal* if A, B is a good frame such that

(i) the union of B-bridges of G not containing A or a_i is maximal,

(ii) subject to (i), the union of fat A-B bridges is maximal,

(iii) subject to (ii), the number of non-fat A-B bridges is minimum.

Lemma 2.2.3 *Suppose A, B is an ideal a_0-frame in γ. Then all A-B bridges are fat.*

Proof. Let M be a non-fat A-B bridge with extreme hands l, r and foot u. Then $V(M \cap A(l, r)) \neq \emptyset$, to avoid the cut $\{l, r, u\}$ in G^*. Note that $M - u - A(l, r)$ has a path from l to r. Hence, by Lemma 2.1.2, $M \cup A[l, r] - u$ contains an induced path P from l to r, such that $M \cup A[l, r] - u - P$ is connected with $A(l, r) \subseteq M \cup A[l, r] - u - P$. Let $A' := A[a_1, l] \cup P \cup A[r, a_2]$. We show that A', B contradicts the choice of A, B.

Clearly, A', B is a good frame, and the union of those B-bridges of G not containing A or a_0 is equal to the union of those B-bridges of G not containing A' or a_0. Moreover, $A(l, r)$ is contained in a non-fat A'-B bridge; otherwise, the union of those fat A'-B bridges properly contains the union of those fat A-B bridges, a contradiction.

Let $M_1, \cdots, M_k$ be the A-B bridges such that for each $i \in [k]$, $M_i \cap A(l, r) \neq \emptyset$, $M_i \neq M$. Then $k \neq 0$; otherwise, G has at least two disjoint edges from $A(l, r)$ to B (as G^* is 6-connected), which contradicts that $A(l, r)$ is contained in a non-fat A'-B bridge.

Since $M_i \cap A(l,r) \neq \emptyset$ for $i \in [k]$, $\bigcup_{i\in[k]} M_i$ and $A(l,r)$ are contained in a same non-fat A'-B bridge; so $M_1, \ldots, M_k$ are non-fat A-B bridges. Now, since $M \cup A[l,r] - u - P$ is connected with $A(l,r) \subseteq M \cup A[l,r] - u - P$, then $\bigcup_{i\in[k]} M_i$ and $M \cup A[l,r] - u - P$ are contained in one single A'-B bridge. Hence, the number of non-fat A'-B bridges is strictly smaller than the number of non-fat A-B bridges, a contradiction. $\square$

Let A, B be a good a_i-frame in γ, let $\{d_1, \cdots, d_t\} = V(B \cap A_i(B))$ with $b_1, d_1, \cdots, d_t$, b_2 on B in order, and let $d_0 = b_1$ and $d_{t+1} = b_2$. For any $i \in [t+1]$, we let J_i^* be the union of $B[d_{i-1}, d_i]$, all the edges between A and $B[d_{i-1}, d_i]$, all those A-B bridges M with $M \cap B \subseteq B[d_{i-1}, d_i]$, and all those B-bridges N of G with $(A + a_i) \cap N = \emptyset$ and $N \cap B \subseteq B[d_{i-1}, d_i]$. Let $u_1, u_2 \in V(A \cap J_i^*)$, such that a_1, u_1, u_2, a_2 occur on A in order with $A[u_1, u_2]$ maximal. Then we say $J_i = G[V(J_i^* \cup A[u_1, u_2])]$ is an A-B *connector*, and u_1, u_2 are the *extreme hands* of J_i. We say that d_{i-1}, d_i are the *feet* of J_i. Note that our definition does not require $J_i \cap J_j = \emptyset$ for $i \neq j$.

An A-B connector J (with feet v_1, v_2 and extreme hands u_1, u_2) is *slim* if $(J - A[u_1, u_2], B[v_1, v_2])$ is planar, and each edge of J with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$ (seen at Figure 2.5). Thus, no slim A-B connector contains an A-B bridge. If J is not a slim connector, we say that J is a *fat* A-B connector (seen at Figure 2.6).

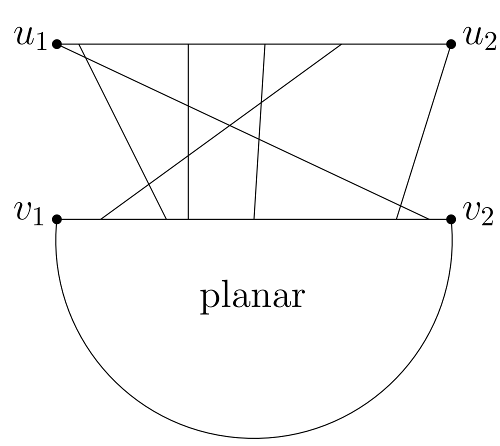

Figure 2.5: A slim connector

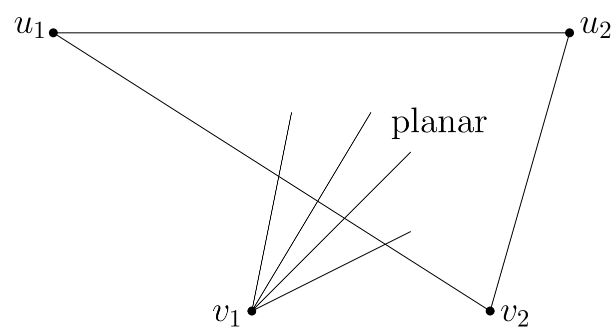

Figure 2.6: A fat connector

Lemma 2.2.4 *Let A, B be an ideal a_0-frame in γ, and J be an A-B connector with feet v_1, v_2 and extreme hands u_1, u_2, such that $V(J)\backslash\{u_1, u_2, v_1, v_2\} \neq \emptyset$. Then*

(i) $u_1 \neq u_2$, there exists a unique $j \in [2]$ such that G has an A-B path from $B[b_j, v_j)$ to $A(u_1, u_2)$, and $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, and

(ii) if J is fat then $N_G(v_j) \cap V(J - v_j - A) \nsubseteq L_p$ for $p \in [2]$, where L_p denotes the subpath of the outer walk of $(J - v_j, A[u_1, u_2], v_{3-j})$ from u_p to v_{3-j} without going through u_{3-p}.

Proof. Since $V(J)\backslash\{u_1, u_2, v_1, v_2\} \neq \emptyset$ and G^* is 6-connected, then $u_1 \neq u_2$ and G has an A-B path from $B - B[b_1, b_2]$ to $A(u_1, u_2)$. By Lemma 2.1.7, there exists a unique $j \in [2]$ such that G has an A-B path from $B[b_j, v_j)$ to $A(u_1, u_2)$.

To prove $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, let T be an A-B path from $t' \in B[b_j, v_j)$ to $t \in A(u_1, u_2)$. If $J - v_j$ contains disjoint paths A^*, B^* from u_1, t to u_2, v_{3-j}, respectively, then $A' := A[a_1, u_1] \cup A^* \cup A[u_2, a_2]$ and $B' := B[b_j, t'] \cup T \cup B^* \cup B[v_{3-j}, b_{3-j}]$ are disjoint paths in $G - v_j - (A_0(B) - B)$ from a_1, b_1 to a_2, b_2, respectively; which contradicts (i) of Lemma 2.2.2. So assume that such A^*, B^* do not exist. Then by Theorem 2.1.1, there exist $m \geq 0$ and a set $\mathcal{D} = \{D_1, \cdots, D_m\}$ of pairwise disjoint nonempty subsets of $V(J - v_j) - \{u_1, u_2, t, v_{3-j}\}$ such that $(J - v_j, \mathcal{D}, u_1, t, u_2, v_{3-j})$ is 3-planar. We choose $D_1, \ldots, D_m$ such that $\bigcup_{i \in [m]} D_i$ is minimal. Then for all $p \in [m]$, $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk (or else, D_p could be chosen to be empty). Obviously, $|D_p| \geq 2$.

Note that $J - v_j - A[u_1, u_2]$ is connected. For, otherwise, let C be a component of $J - v_j - A[u_1, u_2]$ disjoint from $B(v_j, v_{3-j}]$. Then $N_G(C) \subseteq V(A[u_1, u_2]) \cup \{v_j\}$. Since $G - A$ is connected, $v_j \in N_G(C)$; hence, $G[V(C) \cup N_G(C)] - E(A)$ is a non-fat A-B bridge, contradicting Lemma 2.2.3.

If $m = 0$ then $\mathcal{D} = \emptyset$, and $(J - v_j, u_1, t, u_2, v_{3-j})$ is planar; so $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar as $J - v_j - A[u_1, u_2]$ is connected. Hence, $m \geq 1$. Since G^* is 6-connected, for all $p \in [m]$, $N_{J-v_j}(D_p) \cup \{v_j\}$ is not a cut of G separating D_p from other vertices. So $D_p \cap V(A) \neq \emptyset$. Since $D_p \cap \{u_1, u_2, t, v_{3-j}\} = \emptyset$, $|N_{J-v_j}(D_p) \cap A| \geq 2$. Moreover, since A is an induced path and $G[D_p \cup N_{J-v_j}(D_p)]$ does not have a disk representation with $N_{J-v_j}(D_p)$ occurring on the boundary of the disk, $D_p \not\subseteq V(A)$. Thus, $N_{J-v_j}(D_p) \not\subseteq V(A)$ as $J - v_j - A[u_1, u_2]$ is connected. So $|N_{J-v_j}(D_p)| = 3$ and $|N_{J-v_j}(D_p) \cap A| = 2$. Moreover, if we let $\{s_1, s_2, s\} = N_{J-v_j}(D_p)$ such that $s \notin V(A)$ and u_1, s_1, s_2, u_2 occur on A in order, then $J - v_j$ has a path D from s to v_{3-j} disjoint from A; or else, there exists a non-fat A-B bridge with foot v_j, or $G - A$ is not connected. Moreover, since G^* is 6-connected, G has an A-B path R from $r' \in V(B - B[v_1, v_2])$ to $r \in V(A(s_1, s_2))$. By Lemma 2.1.7, $r' \in B[b_j, v_j)$.

Let $H := G[D_p \cup N_{J-v_j}(D_p)]$. If H contains disjoint paths X', R_1 from s_1, r to s_2, s, respectively, then the paths $A' := A[a_1, s_1] \cup X' \cup A[s_2, a_2]$ and $B' := B[b_j, r'] \cup R \cup R_1 \cup D \cup B[v_{3-j}, b_{3-j}]$ in $G - (A_0(B) - B) - v_j$ from a_1, b_1 to a_2, b_2, respectively, contradict Lemma 2.2.2. So such X' and R_1 do not exist. By Lemma 2.1.1, there exist $n \geq 0$ and a set $\mathcal{V} = \{V_1, \cdots, V_n\}$ of pairwise disjoint subsets of D_p such that $(H, \mathcal{V}, s_1, r, s_2, s)$ is 3-planar. However, we see that $\{D_1, \cdots, D_m\} \backslash \{D_p\} \cup \{V_1, \cdots, V_n\}$ contradicts our choice of $\{D_1, \ldots, D_m\}$. This completes the proof of (i).

Next, we prove (ii). Since J contains disjoint paths $A[u_1, u_2]$ and $B[v_1, v_2]$, $N_G(v_j) \cap V(J - v_j - A) \neq \emptyset$. Suppose $N_G(v_j) \cap V(J - v_j - A) \subseteq L_p$ for some $p \in [2]$. Let $u \in N_G[v_j] \cap V(L_p)$, such that $u \neq u_p$, and $L_p[u_p, u]$ is minimal. Since $(J - v_j, A[u_1, u_2], v_{3-j})$ is planar, $J - v_j - A[u_1, u_2]$ is also planar. Let P' denote the subpath of the outer walk of

$J - v_j - A[u_1, u_2]$ from u to v_{3-j} with $P' \subseteq L_p$. Then $N_G(v_j) \cap V(J - v_j - A) \subseteq V(P')$. Let $B' = B[b_j, v_j] \cup \{v_j u\} \cup P' \cup B[v_{3-j}, b_{3-j}]$. Then A, B' is a good frame. The union of those B-bridges of G not containing A and a_0 is contained in the union of those B'-bridges of G not containing A and a_0, which forces $B = B'$ by the choice of A, B. Moreover, by Lemma 2.2.3 and the planarity of $J - v_j$, each edge of J with exactly one end in $A[u_1, u_2]$ has its other end in $B[v_1, v_2]$; so J is a slim connector, a contradiction. □

2.3 Core frames

In this section, we consider the situation when there is a fat connector for some ideal frame in γ (seen at Figure 2.7). The first two lemmas study the structure inside fat connectors, and show that each fat connector has a core in which we can find various disjoint paths.

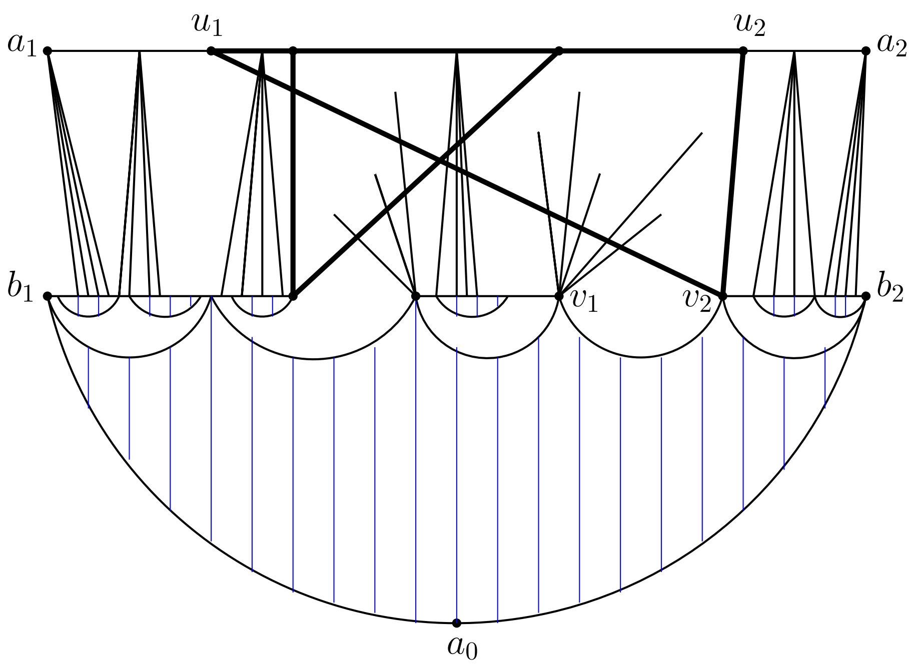

Figure 2.7: An ideal frame with a fat connector

Lemma 2.3.1 *Suppose A, B is an ideal a_0-frame in γ. Let J be a fat A-B connector with feet v_1, v_2 and extreme hands u_1, u_2, such that $(J - v_1, A[u_1, u_2], v_2)$ is planar, a_1, u_1, u_2, a_2*

occur on A in order, b_1, v_1, v_2, b_2 occur on B in order, and G has an A-B path from $A(u_1, u_2)$ to $B[b_1, v_1)$. Then there exists a separation (H, L) in J of order 4 (we allow $H = J$ and L consists of u_1, u_2, v_2 and no edges), such that

(i) $V(H \cap L) = \{v_1, x_1, x_2, y_2\}$, *$u_1, x_1, x_2, u_2$ occur on A in order, v_1, y_2, v_2 occur on B in order, $A[x_1, x_2] \cup B[v_1, y_2] \subseteq H$, and $\{u_1, u_2, v_2\} \subseteq V(L)$;*

(ii) $(L - A, B[y_2, v_2], v_1)$ *is planar, and each edge of L with exactly one end in A has its other end in $V(B[y_2, v_2]) \cup \{v_1\}$;*

(iii) $(H - v_1, A[x_1, x_2], y_2)$ *is planar, $H - v_1 - A[x_1, x_2]$ is connected, $x_1y_2, x_2y_2 \notin E(H)$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from v_1, y_2 to x_1, x_2, respectively, and disjoint paths from v_1, y_2 to x_2, x_1, respectively, and $V(X_1 \cap X_2) = \{y_2\}$ and $N_G(v_1) \cap V(H - A) \not\subseteq V(X_i)$ for $i \in [2]$, where X_i is the path from x_i to y_2 on the outer walk of $H - v_1$ without going through x_{3-i}.*

Proof. Note that by Lemma 2.2.4, if we take $H = J$ and let L consist of u_1, u_2, v_2 and no edges, then (H, L) satisfies (i) and (ii) (with $x_i = u_i$ for $i \in [2]$ and $y_2 = v_2$). Hence, we choose (H, L) satisfying (i) and (ii) and, subject to this, H is minimal. We show that (iii) holds.

Since $(J - v_1, A[u_1, u_2], v_2)$ is planar, $(H - v_1, A[x_1, x_2], y_2)$ is planar. Note that $H - v_1 - A[x_1, x_2]$ is connected; for otherwise, let C be a component of $H - v_1 - A[x_1, x_2]$ not containing y_2, which is also a component of $J - v_1 - A[u_1, u_2]$. Then either it contradicts the definition of frame that $G - A$ is connected, or it contradicts Lemma 2.2.3 that all A-B bridges are fat. By the minimality of H, we see that $x_1y_2, x_2y_2 \notin E(H)$.

For $i = 1, 2$, let X_i denote the path in the outer walk of $H - v_1$ from y_2 to x_i not containing x_{3-i}. Then $V(X_1 \cap X_2) = \{y_2\}$. For, otherwise, H has a separation (H_1, H_2) such that $|V(H_1 \cap H_2)| = 1$, $y_2 \in V(H_1 - H_2)$, and $A[x_1, x_2] \subseteq H_2$. Since G^* is 6-connected, $V(H_1 - H_2) = \{y_2\}$. Let $y_2' \in V(H_1 - y_2)$. Now it is easy to check that the separation $(H - y_2, G[L + y_2'])$ contradicts the choice of (H, L) (that H is minimal).

Next we show that $N_G(v_1) \cap V(H - A) \not\subseteq V(X_i)$ for $i = 1, 2$. For, suppose this is false and, by symmetry, that $N_G(v_1) \cap V(H - A) \subseteq V(X_2)$. Let $y_2' \in N_G(v_1) \cap V(X_2)$ with $X_2[y_2', y_2]$ minimal. Let B' denote the path in the outer walk of $H - A$ from y_2' to y_2 not containing $X_2[y_2', y_2]$. We could choose B so that $B' \subseteq B$. However, this shows that J is not fat, a contradiction.

It remains to show that for $j \in [2]$, $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ contains disjoint paths from v_1, y_2 to x_{3-j}, x_j, respectively. For, otherwise, we may assume by symmetry that $H - A(x_1, x_2) - \{v_1x_1, v_1x_2\}$ does not have disjoint paths from v_1, y_2 to x_1, x_2, respectively. Hence, $H - A(x_1, x_2) - X_2 - \{v_1x_1, v_1x_2\}$ has no path from v_1 to x_1. Since $(H - v_1, A[x_1, x_2], X_2, X_1)$ is planar, there exist $x_1' \in V(A(x_1, x_2))$, $y_2' \in V(X_2)$, and a 2-separation (H_1, H_2) in $H - v_1$ such that $V(H_1 \cap H_2) = \{x_1', y_2'\}$, $x_1, y_2 \in V(H_1)$, $A[x_1', x_2] \subseteq H_2$, and $N_G(v_1) \cap V(H) \subseteq V(H_2 \cup A[x_1, x_2] \cup X_2)$. Then we see that the separation $(H_2, G[H_1 \cup L])$ of J contradicts the choice of (H, L). $\square$

With the notation in Lemma 2.3.1, we say that H is an A-B *core* or a *core* of the fat connector J. Moreover, we say that x_1, x_2 are the *extreme hands* of H, v_1, y_2 are the *feet* of H, and y_2 is the *main foot* of H. For convenience, we write $y_1 := v_1$. By symmetry, we may always assume that a_1, x_1, x_2, a_2 occur on A in order, and that b_1, y_1, y_2, b_2 occur on B in order. Note that $y_1 \in V(A_0(B))$ and G has a path from a_0 to y_1 internally disjoint from B. For $i \in [2]$, let $x_i' \in V(A(x_1, x_2))$ such that x_i', x_i are incident with some finite face of $H - y_1$, and $H - y_1$ has a path from x_i' to y_2 and internally disjoint from A. And for $i \in [2]$, let X_i' be the path from y_2 to x_i' on the outer walk of $H - \{y_1, x_i\}$ without going through x_{3-i}.

Lemma 2.3.2 *Suppose A, B is an ideal a_0-frame, and H is an A-B core with extreme hands x_1, x_2 and feet y_1, y_2, where y_2 is the main foot. Then the degree of y_2 in $H - y_1$ is at least 2 and, for $i \in [2]$, $|V(X_i(x_i, y_2))| \geq 1$ and $V(X_i \cap X_{3-i}') = \{y_2\}$. Moreover, if, for some $i \in [2]$, H does not contain disjoint paths from y_1, y_2 to x_i, x_{3-i}', respectively, and internally disjoint from A, then the following are true:*

(i) No finite face of $H - y_1$ is incident with both y_2 and a vertex of $A(x_1, x_2)$.

(ii) For any $v \in N_G(y_1) \cap V(H)$ with $v \notin X'_{3-i} \cup A(x_i, x_{3-i}]$, there exist $c_1 \in A(x_i, x'_{3-i})$ and $c_2 \in X'_{3-i}(x'_{3-i}, y_2)$, such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_{3-i}\}$ separating v from x_i, and there exist internally disjoint paths from v to c_1, c_2 in $H - \{y_1, x_{3-i}\}$, respectively, which are internally disjoint from $X'_{3-i} \cup A[x_i, x'_{3-i}]$.

(iii) H has disjoint paths from y_1, y_2 to x_{3-i}, x'_i, respectively, and internally disjoint from A.

Proof. By Lemma 2.3.1, $V(X_1 \cap X_2) = \{y_2\}$ and $x_1y_2, x_2y_2 \notin E(H)$; so the degree of y_2 in $H - y_1$ is at least 2 and $|V(X_i(x_i, y_2))| \geq 1$. Moreover, $V(X_i \cap X'_{3-i}) = \{y_2\}$ for $i \in [2]$; for, suppose there exists $c \in V(X_i \cap X'_{3-i}) - \{y_2\}$, then $\{c, y_1, y_2, x_{3-i}\}$ is a cut in G separating $V(X_{3-i})$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By symmetry, we may assume that H does not contain disjoint paths from y_1, y_2 to x_1, x'_2, respectively, that are internally disjoint from A.

To prove (i), suppose there exists $v_0 \in V(A(x_1, x_2))$ such that v_0, y_2 are incident with some finite face in $H - y_1$. Since $(H - y_1, A[x_1, x_2], y_2)$ is planar, $H - y_1$ has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{y_2, v_0\}$, $X_1 \subseteq H_1$, and $X_2 \subseteq H_2$. Now, we further choose v_0 so that H_1 is minimal.

Now, we see that H_2 contains a path P_2 from y_2 to x'_2 and internally disjoint from A; for otherwise, $V(H_2 \cap A) = \{x_2\}$ and, hence, $\{y_1, y_2, x_2\}$ is a cut in G^* separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, let P_1 be the path from y_1 to x_1 in $H - V(A(x_1, x_2]) \cup \{y_2\}$ (by (iii) of Lemma 2.3.1). Since $v_0 \neq x_1$, $V(P_1 \cap H_2) = \emptyset$, and so $V(P_1 \cap P_2) = \emptyset$. However, the existence of P_1, P_2 contradicts that H does not contain disjoint paths from y_1, y_2 to x_1, x'_2, respectively, and internally disjoint from A. This completes the proof of (i).

To prove (ii), let $v \in N_G(y_1) \cap V(H)$ such that $v \notin X'_2 \cup A(x_1, x_2]$. Since $(H - \{y_1, x_2\}, A[x_1, x'_2] \cup X'_2[x'_2, y_2])$ is planar and $H - y_1 - A(x_1, x_2] \cup X'_2$ does not have a path from v to x_1, there exist $c_1, c_2 \in V(A(x_1, x_2'] \cup X_2')$ such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_2\}$ separating v from x_1. We may assume c_1, c_2 occur on $A(x_1, x_2'] \cup X_2'[x_2', y_2]$ in order.

Note that $c_1 \notin V(X_2')$, to avoid the cut $\{c_1, c_2, y_1, x_2\}$ in G^*. Moreover, $c_2 \notin A(x_2', y_2]$; or else, $H - V(A) \cup \{y_1\}$ is not connected, contradicting (iii) of Lemma 2.3.1.

We choose c_1, c_2 such that $A[c_1, x_2']$ and $X_2'[x_2', c_2]$ are minimal. Then $H - \{y_1, x_2\}$ contains internally disjoint paths from v to c_1, c_2, respectively, and internally disjoint from $A \cup X_2'$. Moreover, by (i), $c_2 \neq y_2$. This completes the proof of (ii).

To prove (iii), observe that $V(X_1' \cap X_2') = \{y_2\}$. For otherwise, let $c \in V(X_1' \cap X_2')$ with $c \neq y_2$. Since y_2 has degree at least 2 in $H - y_1$ and $x_1 y_2, x_2 y_2 \notin E(H)$, $\{x_1, x_2, y_1, y_2, c\}$ is a cut in G^* separating $V(X_1 \cup X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now, let $u_2 \in V(X_2 \cap X_2')$ such that $X_2[x_2, u_2]$ is minimal. Moreover, let $v \in N_G(y_1) \cap V(H - A)$. If $v \in V(X_2')$ then let $P_2 = v = c_2$; and if $v \notin V(X_2')$ then by (ii), there exist $c_1 \in V(A(x_1, x_2'))$ and $c_2 \in V(X_2'(x_2', y_2))$, such that $\{c_1, c_2\}$ is a cut in $H - \{y_1, x_2\}$ separating v from x_1, and there exists a path P_2 from v to c_2 in $H - \{y_1, x_2\}$, which is internally disjoint from $X_2' \cup A[x_1, x_2']$. Since $V(X_1' \cap X_2') = \emptyset$ and $(H - y_1, A[x_1, x_2] \cup X_2)$ is planar, P_2 is disjoint from X_1'. Now, X_1' and $y_1 v \cup P_2 \cup X_2'[c_2, u_2] \cup X_2[u_2, x_2]$ are disjoint paths from y_2, y_1 to x_1', x_2, respectively, in H, which are internally disjoint from A. $\square$

The next lemma describes interactions between cores from different connectors and finds a path B' so that A, B' is a good frame in γ which will eventually be used to form a special frame A', B' in γ.

Lemma 2.3.3 *Let A, B be an ideal a_0-frame in γ, and let H^j, $j \in [m]$, be the A-B cores in γ such that H^j has extreme hands x_1^j, x_2^j and feet y_1^j, y_2^j. Then*

(i) for any distinct $i, j \in [m]$, $A[x_1^i, x_2^i] \subseteq A[x_1^j, x_2^j]$ or $A[x_1^j, x_2^j] \subseteq A[x_1^i, x_2^i]$,

(ii) for any $j \in [m]$, $H^j - A[x_1, x_2]$ has a path P_j from y_1^j to y_2^j such that $|V(P_j)| \geq 3$, $H^j - P_j$ is connected, and P_j is induced in $G - y_1^j y_2^j$,

(iii) A, B' *is a good* a_0*-frame and* $A_0(B') = A_0(B)$*, where* B' *is obtained from* B *by replacing* $B[y_1^j, y_2^j]$ *with the path* P_j *in (ii) for* $j \in [m]$*, and*

(iv) with G_0' *as the graph obtained from* G *by deleting the component of* $G - B'$ *containing* A*,* $(G_0', a_0, b_1, B', b_2)$ *is planar and, for any* $v \in B'(y_1^j, y_2^j)$*, the degree of* v *in* G_0' *is 2.*

Proof. To prove (i), assume for some distinct $i, j \in [m]$ with $i \neq j$, we have $A[x_1^i, x_2^i] \nsubseteq A[x_1^j, x_2^j]$, and $A[x_1^j, x_2^j] \nsubseteq A[x_1^i, x_2^i]$. Without loss of generality, let $b_1, y_1^i, y_2^i, y_1^j, y_2^j, b_2$ occur on B in this order, and a_1, x_1^i, x_2^j, a_2 occur on A in this order with $x_2^i, x_1^j \in A(x_1^i, x_2^j)$. By Lemma 2.3.1, $H^i - A(x_1^i, x_2^i)$ has two disjoint A-B paths P_1, P_2 from y_1^i, y_2^i to x_2^i, x_1^i, respectively, and $H^j - A(x_1^j, x_2^j)$ has two disjoint A-B paths P_3, P_4 from y_1^j, y_2^j to x_2^j, x_1^j, respectively. Therefore, P_1, P_2, P_3, P_4 form a double cross in A, B, a contradiction.

For (ii), let $j \in [m]$. Since H^j is a core, $H^j - y_1^j y_2^j - A$ has a path T_j from y_1^j to y_2^j. So by Lemma 2.1.2, $H^j - y_1^j y_2^j$ has an induced path P_j from y_1^j to y_2^j such that $H^j - y_1^j y_2^j - P_j$ is connected and $A[x_1^j, x_2^j] \subseteq H^j - y_1^j y_2^j - P_j$.

To see (iii), we observe that $A_0(B')$, the B'-bridge of G containing a_0, is the same as, $A_0(B)$, the B-bridge of G containing a_0. So A, B' is also a good a_0-frame.

To prove (iv), let C denote the component of $G - B'$ containing A; so $G_0' = G - C$. By Lemma 2.1.6, $(A_0(B'), a_0, b_1, B', b_2)$ is planar. Thus, to show that $(G_0', a_0, b_1, B', b_2)$ is planar, it suffices to show that for any A-B connector J with feet v_1, v_2, $(J - C, B'[v_1, v_2])$ is planar. This is clear when J is a slim connector. So assume J is a fat connector. Then J has a separation (H, L) satisfying (i), (ii), and (iii) of Lemma 2.3.1. By (ii) of Lemma 2.3.1, $(L - A, B' \cap L)$ is planar. Since $H - B' \subseteq C$, we see that $(J - C, B'[v_1, v_2])$ is planar.

Moreover, for any $v \in B'(y_1^j, y_2^j)$, since $B'[y_1^j, y_2^j]$ is a path in the core H^j, then, combined with (ii) that P_j is induced in $G - y_1 y_2$, the degree of v in G_0' is exactly 2. $\square$

In the remaining parts of this section, suppose A, B is an ideal frame in γ. By (i) of Lemma 2.3.3, there exists an A-B core H with extreme hands x_1, x_2 and feet y_1, y_2 (y_2 as the main foot), which is also an A-B' core, such that for any core H^j with extreme hands x_1^j, x_2^j, we have $A[x_1^j, x_2^j] \subseteq A[x_1, x_2]$. We call such a core H a *main* A-B' core or a *main* A-B core. We also use B' to denote the path in (iii) of Lemma 2.3.3 and G_0' to denote the graph in (iv) of Lemma 2.3.3. By (iii) of Lemma 2.3.2, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths in $H - A(x_1, x_2)$ from x_1, x_2 to y_i, y_{3-i}, respectively.

We consider the structure of G outside H. Let $r_1 \in V(B'[b_1, y_1))$, such that $B'[b_1, r_1)$ contains no foot of A-B' cores in γ, G has no A-B' path from $A(x_1, x_2)$ to $B'[b_1, r_1)$, and subject to these conditions, $B'[b_1, r_1]$ is maximal. Then G has a path R_1 from r_1 to some $r \in V(A(x_1, x_2))$ and internally disjoint from A such that $R_1 = r_1 r$ or R_1 is contained in some A-B' core H' with r_1 as a foot and does not contain the other foot of H'.

For notational convenience, we let $t_1 := r_1$ and $t_2 := y_2$. We derive useful structure of G outside $A[x_1, x_2] \cup B'[t_1, t_2]$.

Lemma 2.3.4 *G has no A-B' path from $A(x_1, x_2)$ to $B' - B'[t_1, t_2]$ or from $B'(t_1, t_2)$ to $A - A[x_1, x_2]$.*

Proof. By the maximality of $B'[b_1, r_1]$, G has no A-B' path from $A(x_1, x_2)$ to $B'[b_1, t_1)$. Since no double cross exists in A, B (by Lemma 2.1.7), G has no A-B' path from $A(x_1, x_2)$ to $B'(t_2, b_2]$. Moreover, G has no A-B' path from $B'(t_1, t_2)$ to $A[a_1, x_1) \cup A(x_2, a_2]$; to avoid forming a double cross in A, B with R_1 and one of $\{P_{1,2}, P_{2,1}\}, \{P_{1,1}, P_{2,2}\}$. $\square$

Lemma 2.3.5 *Let $e_3 = a_3 b_3, e_4 = a_4 b_4 \in E(G)$ with $a_3, a_4 \in V(A)$ and $b_3, b_4 \in V(B')$.*

(i) If for some $i \in [2]$, $a_3 \in V(A[a_i, x_i))$, $b_3 \in V(B'[t_2, b_2))$, $a_4 \in V(A(a_3, x_i])$, and $b_4 \in V(B'[b_1, t_1))$, then G_0' has a 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \in B'[b_1, b_4]$ and $b_2' \in B'[t_2, b_3]$, which separates $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$ in G_0'.

(ii) If for some $i \in [2]$, $a_3 \in V(A[a_i, x_i))$, $b_3 \in V(B'(b_1, t_1])$, $a_4 \in V(A(a_3, x_i])$, and $b_4 \in V(B'(t_2, b_2])$, then one of the following holds:

(a) *G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, t_1]$ and $b'_2 \in B'[b_4, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0;*

(b) *G'_0 has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_4, b_2]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.*

(iii) *If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(b_1, t_1))$, then G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_3, b_4]$ and $b'_2 \in B'[t_2, b_2]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.*

(iv) *If $a_3 \in V(A[a_1, x_1])$, $a_4 \in V(A[x_2, a_2])$, and $b_3, b_4 \in V(B'(t_2, b_2))$, then one of the following holds:*

(a) *G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, t_1]$ and $b'_2 \in B'[b_3, b_4]$, which separates $B'[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0;*

(b) *G'_0 has a 2-cut $\{y_1, b'_2\}$ with $b'_2 \in B'[b_3, b_4]$, which separates $B'[y_1, b'_2]$ from $\{a_0, b_1, b_2\}$ in G'_0.*

Proof. Suppose (i) fails. Then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, there exist disjoint paths B'_2, A'_0 in $G'_0 - (B'[b_1, b_4] \cup B'[y_2, b_3])$ from b_2, a_0 to y_1, r_1, respectively. Now, $A[a_i, a_3] \cup e_3 \cup B'[y_2, b_3] \cup P_{3-i,2} \cup A(x_i, a_{3-i}] \cup R_1 \cup A'_0$ and $B'[b_1, b_4] \cup e_4 \cup A[a_4, x_i] \cup P_{i,1} \cup B'_2$ show that γ is feasible, a contradiction.

Now suppose (ii) fails. Then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G'_0 - (B'[b_3, r_1] \cup B'[b_4, b_2])$ contains two disjoint paths B^*_1, A^*_0 from b_1, a_0 to y_1, y_2, respectively. Now $A[a_i, a_3] \cup e_3 \cup B'[b_3, r_1] \cup R_1 \cup A(x_i, a_{3-i}] \cup P_{3-i,2} \cup A^*_0$ and $B^*_1 \cup P_{i,1} \cup A[a_4, x_i] \cup e_4 \cup B'[b_4, b_2]$ show that γ is feasible, a contradiction.

If (iii) fails then, since $(G'_0, a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G'_0 - (B'[b_3, b_4] \cup B'[t_2, b_2])$ has disjoint paths B^*_1, A^*_0 from b_1, a_0 to r_1, y_1, respectively. Moreover, by Lemma 2.3.2, for some $p \in [2]$, H contains disjoint paths Y_1, Y_2 from x_p, x'_{3-p} to y_1, y_2, respectively. Thus, $A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A^*_0$ and $B^*_1 \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[t_2, b_2]$ show that γ is feasible, a contradiction.

Finally, suppose (iv) fails. Then, since $(G_0', a_0, b_1, B', b_2)$ is planar and y_2 is the main foot of H, $G_0' - (B'[b_1, t_1] \cup B'[b_3, b_4])$ has disjoint paths B_2', A_0' from b_2, a_0 to y_2, y_1, respectively. Thus, $A[a_1, x_1] \cup e_3 \cup B'[b_3, b_4] \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A_0'$ and $B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B_2'$ show that γ is feasible, a contradiction. $\square$

Lemma 2.3.6 *G_0' does not have 3-cuts $\{a_0', b_1', b_2\}$ and $\{a_0'', b_1, b_2''\}$ with $b_1' \in V(B'(b_1, t_1])$ and $b_2'' \in V(B'[t_2, b_2))$ such that $\{a_0', b_1', b_2\}$ separates $B'[b_1', b_2]$ from $\{a_0, b_1, b_2\}$ and $\{a_0'', b_1, b_2''\}$ separates $B'[b_1, b_2'']$ from $\{a_0, b_1, b_2\}$.*

Proof. For, suppose both 3-cuts exist. We choose $\{a_0', b_1', b_2\}$ with $B'[b_1, b_1']$ minimal, and choose $\{a_0'', b_1, b_2''\}$ with $B'[b_2'', b_2]$ minimal. Then, since G_0' has a path from a_0 to y_1 and internally disjoint from B', it follows from Lemma 2.1.8 that

(1) (ii) or (iii) or (iv) of Lemma 2.1.8 holds (and so $c(A, B') \geq 1$).

By the minimality of $B[b_1, b_1']$ and $B[b_2'', b_2]$, it follows from (1) and planarity of $(G_0', a_0, b_1, B', b_2)$ that

(2) $G_0' - B'(b_1, b_1') - B'(b_2'', b_2)$ has disjoint paths B_1^*, B_2^*, A_0^* from b_1, b_2, a_0 to b_1', b_2'', y_1, respectively, which are internally disjoint from B'.

Also by the minimality of $B[b_1, b_1']$ and $B[b_2'', b_2]$, it follows from (iii) and (iv) of Lemma 2.3.5 and Lemmas 2.1.8 and 2.1.9 that

(3) G has no edge from $B'(b_1, b_1')$ to $A[a_1, x_1]$ or no edge from $B'(b_1, b_1')$ to $A[x_2, a_2]$; and G has no edge from $B'(b_2'', b_2)$ to $A[a_1, x_1]$ or no edge from $B'(b_2'', b_2)$ to $A[x_2, a_2]$.

Next, we claim that

(4) $\alpha(A, B') \leq 1$.

For, suppose $\alpha(A, B') = 2$. Then, by (1), $a_0 = a_0' = a_0''$; so $c(A, B') \geq 2$. For convenience, let $s_1 := b_1'$ and $s_2 := b_2''$. Now, since $\alpha(A, B') = 2$, G_0' has a path A_i^* (for each $i \in [2]$)

from a_0 to b_i and internally disjoint from B'. Hence, since G^* is 6-connected, $B'(b_i, s_i) \neq \emptyset$ for $i \in [2]$.

We claim that there do not exist $e = ab, e' = a'b' \in E(G)$, such that for some $i \in [2]$, $a, a' \in A(a_i, x_i)$, $b \in B'[b_1, s_1)$, and $b' \in B'(s_2, b_2]$. For, otherwise, $\alpha(A, B') = 2$ and $c(A, B') = 0$ by Lemma 2.2.1, because of the path $B'[b_1, b] \cup e \cup A[a, a'] \cup e' \cup B'[b', b_2]$ from b_1 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A[x_i, x_{3-i}) \cup P_{i,2} \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. This is a contradiction.

Since G^* is 6-connected, G has at least three pairwise disjoint edges from $B'(b_i, s_i)$ (for each $i \in [2]$) to $A[a_1, x_1] \cup A[x_2, a_2]$. By (3), for each $i \in [2]$, we may assume for some $j \in [2]$, G has no edge from $B'(b_i, s_i)$ to $A[a_j, x_j]$. Now, by symmetry, we assume G has no edge from $B'(b_1, s_1)$ to $A[x_2, a_2]$.

By Lemma 2.1.7, G has no cross from $A[a_1, x_1]$ to $B'(b_1, s_1)$. So, let $f_i = u_i v_i$ for $i \in [3]$ be pairwise disjoint edges of G with $u_i \in A[a_1, x_1]$ and $v_i \in B'(b_1, s_1)$, such that a_1, u_1, u_3, u_2, a_2 occur on A in order, and b_1, v_1, v_3, v_2, b_2 occur on B' in order. We choose f_1, f_2 so that $A[u_1, u_2] \cup B'[v_1, v_2]$ is maximal.

Then G has no edge from $B'(s_2, b_2)$ to $A[a_1, x_1]$. For otherwise, G has no edge from $B'(s_2, b_2)$ to $A[x_2, a_2]$ and, hence, has at least three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[a_1, x_1]$. Therefore, G has an edge from $A(a_1, x_1)$ to $B'(s_2, b_2)$, which together with f_3 contradicts our claim above.

Thus, G has three pairwise disjoint edges from $B'(s_2, b_2)$ to $A[x_2, a_2]$. Since G has no cross from $A[x_2, a_2]$ to $B'(s_2, b_2)$ (by Lemma 2.1.7), we let $f_j = u_j v_j$ for $j \in \{4, 5, 6\}$ be pairwise disjoint edges of G with $u_j \in A[x_2, a_2]$ and $v_j \in B'(s_2, b_2)$, such that a_1, u_4, u_6, u_5, a_2 occur on A in order, and b_1, v_4, v_6, v_5, b_2 occur on B' in order. Choose f_4, f_5 so that $A[u_4, u_5] \cup B'[v_4, v_5]$ is maximal.

Now by the maximality of $A[u_1, u_2]$, G has an edge $f_7 = u_7 v_7$ with $u_7 \in A(u_1, u_2)$ and $v_7 \in B'[t_2, b_2]$, to avoid the cut $\{u_1, u_2, b_1, s_1, a_0\}$ in G^*. Similarly, by the maximality of $A[u_4, u_5]$, G has an edge $f_8 = u_8 v_8$ with $u_8 \in A(u_4, u_5)$ and $v_8 \in B'[b_1, t_1]$. Now, by the claim above, $v_7 \in B'[t_2, s_2]$ and $v_8 \in B'[s_1, t_1]$. Hence, f_2, f_4, f_7, f_8 form a double cross, contradicting Lemma 2.1.7. $\square$

For $i \in [2]$, let $a_i' \in V(A[a_i, x_i])$ with $A[a_i, a_i']$ minimal such that $a_i' = x_i$ or G has an edge from a_i' to $B'(b_1', b_2)$. Then G has an edge $e_4 = a_4b_4$ with $a_4 \in A(a_1', x_1] \cup A[x_2, a_2')$ and $b_4 \in B[b_1, b_1')$; for, otherwise, $\{a_0, a_1', a_2', b_1', b_2\}$ would be a 5-cut in G^* separating H from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. By symmetry, we may assume

(5) $a_4 \in A(a_1', x_1]$.

Let $e_3 = a_3b_3 \in E(G)$ with $a_3 = a_1'$ and $b_3 \in B'(b_1', t_1] \cup B'[t_2, b_2)$. Since e_3, e_4 and the paths in H do not form a double cross (by Lemma 2.1.7), we have

(6) $b_3 \in B'[t_2, b_2)$.

Let $e = ab \in E(G)$ with $a \in A[a_1, a_3]$ and $b \in B'[b_3, b_2]$, such that $B'[b, b_2]$ is minimal, and subject to this, $A[a_1, a]$ is minimal. Further, let $e' = a'b' \in E(G)$ with $a' \in A[a_1, a_4]$ and $b' \in B'[b_1, b_4]$, such that $B'[b_1, b']$ is minimal, and subject to this, $A[a_1, a']$ is minimal.

Similarly, for each $i \in [2]$, let $a_i'' \in V(A[a_i, x_i])$ with $A[a_i, a_i'']$ minimal such that $a_i'' = x_i$ or G has an edge from a_i'' to $B'(b_1, b_2'')$. Since G^* is 6-connected, there exist $j \in [2]$ and $e_6 = a_6b_6 \in E(G)$ such that $a_6 \in A(a_j'', x_j]$ and $b_6 \in B'(b_2'', b_2]$. Since $a_j'' \neq x_j$, it follows from Lemma 2.1.7 that there exists $e_5 = a_5b_5 \in E(G)$ such that $a_5 = a_j''$ and $b_5 \in B'(b_1, t_1]$.

(7) $b \in B'(b_2'', b_2]$.

For, otherwise, $b \notin B'(b_2'', b_2]$. Then, $j = 2$ and $a_6 \in A[x_2, a_2'')$ by the choice of e. Hence, $b_5 \in B'[b_1, b_4]$ to avoid the double cross e_3, e_4, e_5, e_6. So $b_5 = b_1$ by (3), a contradiction to $b_5 \in B'(b_1, t_1]$. $\square$

If $a' \neq x_1$ then $\alpha(A, B') = 2$ by Lemma 2.2.1 and the following paths: the path $A[a_1, a'] \cup e' \cup B'[b_1, b']$ from a_1 to b_1, the path $A[a_1, a] \cup e \cup B'[b, b_2]$ from a_1 to b_2, the path $B_1^* \cup B'[b_1', r_1] \cup R_1 \cup A[x_1, x_2) \cup P_{1,2} \cup B'[y_2, b_2''] \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{2,1} \cup A[x_2, a_2]$ from a_0 to a_2. This contradicts (4).

So $a' = x_1$. Hence, by the choice of e' and Lemma 2.1.7, G has no edge from $A[a_1, x_1)$ to $B'[b_1, t_1]$. Thus, G has an edge from a_1 to $B'[t_2, b_2]$. So by the choice of e and by Lemma 2.1.7, $a = a_1$ and, hence, $b \neq b_2$.

We claim $a_6 \in A[x_2, a_2'')$. For, otherwise, $a_6 \in A(a_1'', x_1]$. Then $a_5 \in A[a_1, x_1)$. Now, e_5 contradicts the choice of e', or $e_5, e', P_{1,2}, P_{2,1}$ form a double cross, contradicting Lemma 2.1.7.

Thus, by (3), $b_6 = b_2$. Moreover, $b_5 \in B'[b_1, b']$ to avoid the double cross e, e', e_5, e_6. Now, by (3), we may further assume $b_5 = b_1$, a contradiction to $b_5 \in B'(b_1, t_1]$. □

Lemma 2.3.7 *Let $\{a_0', b_1', b_2'\}$ be a cut in G_0' separating $B'[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, with $b_1' \in B'[b_1, t_1]$ and $b_2' \in B[t_2, b_2]$. Then $b_1' = b_1$, $b_2' \neq b_2$, $a_0' = a_0$, y_1 is a cut vertex in G_0' separating b_2 from $\{a_0, b_1\}$, b_2 has degree 1 in G_0', and for some $p \in [2]$, G has an edge from b_2 to x_p and no edge from b_2 to $A - x_p$.*

Proof. For $i \in [2]$, let $a_i' \in V(A[a_i, x_i])$ with $A[a_i, a_i']$ minimal such that $a_i' = x_i$ or G has an edge from a_i' to $B'(b_1', b_2')$. Since G^* is 6-connected, there exist $i, j \in [2]$ such that G has an edge $e_4 = a_4 b_4$ with $a_4 \in A(a_i', x_i]$ and $b_4 \in B'[b_j, b_j')$. By symmetry, assume $i = 1$. Then $a_1' \neq x_1$ and let $e_3 = a_3 b_3 \in E(G)$ such that $a_3 = a_1'$ and $b_3 \in B'(b_1', t_1] \cup B'[t_2, b_2')$. Now $b_3 \in B'[t_{3-j}, b_{3-j}')$, to avoid the double cross formed by e_3, e_4 and two paths in H (by Lemma 2.1.7).

First, we show that

(1) $b_1' = b_1$.

For, suppose $b_1' \neq b_1$. Choose the 3-cut $\{a_0', b_1', b_2'\}$ with $b_1' \neq b_1$, such that $B[b_2', b_2]$ is minimal and, subject to this, $B[b_1, b_1']$ is minimal.

Observe that $b_4 \in B[b_1, b_1')$. For, otherwise, $b_4 \in B(b_2', b_2]$. Then $b_3 \in B(b_1', t_1]$. Now, by Lemma 2.1.9 and (ii) of Lemma 2.3.5, G_0' has a 3-cut contradicting the choice of

$\{a_0', b_1', b_2'\}$.

Then $b_3 \in B'[t_2, b_2')$. Hence, because of e_3, e_4, it follows from (i) of Lemma 2.3.5 that G_0' has a 3-cut $\{a_0'', b_1'', b_2''\}$ with $b_1'' \in B'[b_1, b_4]$ and $b_2'' \in B'[t_2, b_3]$, separating $B'[b_1'', b_2'']$ from $\{a_0, b_1, b_2\}$. By Lemma 2.1.8 and the choice of $\{a_0', b_1', b_2'\}$, we have $b_1'' = b_1$.

By Lemma 2.3.6, $b_2' \neq b_2$. Hence, by Lemma 2.1.8, there exists $a_0^* \in V(G_0')$, such that $\{b_1'', b_2', a_0^*\}$ is a 3-cut in G_0' separating $\{a_0, b_1, b_2\}$ from $B'[b_1'', b_2']$. For $i \in [2]$, let $a_i'' \in A[a_i, x_i]$ with $A[a_i, a_i'']$ minimal such that $a_i'' = x_i$ or G has an edge from a_i'' to $B'(b_1'', b_2')$.

Since G^* is 6-connected, there exist $k \in [2]$ and $e_5 = a_5 b_5 \in E(G)$ with $a_5 \in A(a_k'', x_k]$ and $b_5 \in B'(b_2', b_2]$. Let $e_6 = a_6 b_6 \in E(G)$ with $a_6 = a_k''$ and $b_6 \in B'(b_1'', t_1] \cup B'[t_2, b_2')$. Then $b_6 \in B'(b_1'', t_1]$, to avoid the double cross formed by e_5, e_6 and two paths in H. Because of e_5 and e_6, it follows from (ii) of Lemma 2.3.5 and the choice of $\{a_0', b_1', b_2'\}$ that G_0' has a 2-cut $\{y_1, b_2^*\}$ with $b_2^* \in B'[b_5, b_2]$, separating $B'[y_1, b_2^*]$ from $\{a_0, b_1, b_2\}$. But then, by Lemma 2.1.9, $\{y_1, b_2^*\}$ and $\{a_0', b_1', b_2'\}$ force a 3-cut in G_0', which contradicts the choice of $\{a_0', b_1', b_2'\}$. $\Box$

Since G^* is 6-connected, it follows from (1) that $b_2 \neq b_2'$. We choose $\{a_0', b_1', b_2'\}$ so that $B[b_2, b_2']$ is minimal. Then, by (1) and (ii) of Lemma 2.3.5, G_0' has a 2-cut $\{y_1, b_2''\}$ with $b_2'' \in B'[b_4, b_2]$, separating $B'[y_1, b_2'']$ from $\{a_0, b_1, b_2\}$.

Moreover, $b_2'' = b_2$; for, otherwise, by Lemma 2.1.9, $\{y_1, b_2''\}$ and $\{a_0', b_1', b_2'\}$ force a 3-cut in G_0', which contradicts the choice of $\{a_0', b_1', b_2'\}$. Hence, y_1 is a cut vertex in G_0' separating b_2 from $\{a_0, b_1\}$ and $\alpha(A, B') \leq 1$. And (for any choice of $\{a_0', b_1', b_2'\}$,) $a_0' = a_0$; or else, since y_1 is a cut vertex in G_0' separating b_2 from $\{a_0, b_1\}$, $\{b_1, a_0', b_2', b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction.

So by (1), $G_0' - B'(b_1, t_1) \cup B'(y_1, b_2]$ has disjoint paths B_1^*, A_0^* from b_1, a_0 to t_1, y_1, respectively, such that A_0^* is internally disjoint from B'. By the choice of $\{a_0', b_1', b_2'\}$, $G_0' - B'(b_2', b_2)$ has a path B_2^* from b_2 to b_2'.

(2) For $i \in [2]$, if G has an edge from $B'(b_2', b_2]$ to $A[a_i, x_i)$, then G has no edge from

$A[a_i, x_i)$ to $B'[b_1, t_1)$.

For, suppose for some $i \in [2]$, G has an edge e from $b \in B'(b'_2, b_2]$ to $a \in A[a_i, x_i)$ and an edge e' from $a' \in A[a_i, x_i)$ to $b' \in B'[b_1, t_1)$.

Then, $\alpha(A, B') = 2$, by Lemma 2.2.1 and the following paths: $A[a_i, a'] \cup e' \cup B'[b_1, b']$ from a_i to b_1, the path $A[a_i, a] \cup e \cup B'[b, b_2]$ from a_i to b_2, the path $B_1^* \cup R_1 \cup A[x_i, x_{3-i}) \cup P_{i,2} \cup B_2^*$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. This is a contradiction. $\square$

(3) $B'(b'_2, b_2) = \emptyset$, and so b_2 has degree 1 in G'_0.

For, suppose $B'(b'_2, b_2) \neq \emptyset$. Then, as G^* is 6-connected, G has edges from $B'(b'_2, b_2)$ to $A[a_1, x_1] \cup A[x_2, a_2]$.

Indeed, G has an edge e_3 from $B'(b'_2, b_2)$ to $A[a_1, x_1]$, and an edge e_4 from $B'(b'_2, b_2)$ to $A[x_2, a_2]$. For otherwise, there exists $i \in [2]$, such that all edges of G from $B'(b'_2, b_2)$ to A end in $A[a_i, x_i]$. Let $u_1, u_2 \in V(A[a_i, x_i])$, such that G has edges from $B'(b'_2, b_2)$ to u_1, u_2, respectively, and, subject to this, $A[u_1, u_2]$ is maximal. Now, by Lemma 2.1.7, G has no edge from $A(u_1, u_2)$ to $B'[t_2, b'_2)$. Moreover, by (2), G has no edge from $A(u_1, u_2)$ to $B'[b_1, t_1)$. But then, $\{t_1, u_1, u_2, b'_2, b_2\}$ is a cut in G separating $V(A[u_1, u_2] \cup B'[b'_2, b_2])$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now $A[a_1, x_1] \cup e_3 \cup B'(b'_2, b_2) \cup e_4 \cup A[x_2, a_2] \cup Y_1 \cup A_0^*$ and $B'[b_1, r_1] \cup R_1 \cup A(x_1, x_2) \cup Y_2 \cup B'[y_2, b'_2] \cup B_2^*$ show that γ is feasible, a contradiction. $\square$

(4) G has no edge from b_2 to $A[a_1, x_1) \cup A(x_2, a_2]$.

Suppose for some $i \in [2]$, G has an edge e from b_2 to $a \in A[a_i, x_i)$. Let $e' = a_1 b' \in E(G)$ with $b' \neq t_1$. Obviously, $b' \notin B'[t_2, b_2)$; otherwise, e, e' and two disjoint paths in H force a double cross, contradicting Lemma 2.1.7.

So $b' \in B[b_1, t_1)$. Now $\alpha(A, B') = 2$ by Lemma 2.2.1 and the following paths: the path $e' \cup B'[b_1, b']$ from a_i to b_1, the path $A[a_i, a] \cup e$ from a_i to b_2, the path $B_1^* \cup R_1 \cup$

$A[x_i, x_{3-i}) \cup P_{i,2} \cup B'[y_2, b_2]$ from b_1 to b_2, and the path $A_0^* \cup P_{3-i,1} \cup A[x_{3-i}, a_{3-i}]$ from a_0 to a_{3-i}. However, this is a contradiction. □

Now, since the degree of b_2 in G is at least 2, it follows from (4) that G has an edge from b_2 to x_p for some $p \in [2]$. If G has no edge from b_2 to x_{3-p} then we are done. So assume $b_2x_1, b_2x_2 \in E(G)$. Then $a_1 \neq x_1$ and $a_2 \neq x_2$. Now, by Lemma 2.1.7, G has no edge from $\{a_1, a_2\}$ to $B'[t_2, b_2)$. Since G^* is 6-connected, G has edges e_1, e_2 from $B'(b_1, t_1)$ to a_1, a_2, respectively. But then, it follows from (iii) of Lemma 2.3.5 that G'_0 contains a 3-cut, which contradicts (1). □

Lemma 2.3.8 *H is the unique main A-B' core in γ.*

Proof. Suppose for a contradiction that H'' is a main A-B' core with $H'' \neq H$, and let w_1, w_2 be the feet of H'' (with w_2 as the main foot). Then, by Lemma 2.1.7, $w_2 = r_1$ and $b_1, w_2, w_1, y_1, y_2, b_2$ occur on B' in order.

Recall the definition of x'_i, X'_i before Lemma 2.3.2. For $i \in [2]$, let $x''_i \in V(A(x_1, x_2))$ such that x''_i, x_i are incident with some finite face of $H'' - w_1$, and $H'' - w_1$ has a path from x''_i to w_2 and internally disjoint from A. So for $i \in [2]$, let X''_i be the path from w_2 to x''_i on the outer walk of $H'' - \{w_1, x_i\}$ without going through x_{3-i}, and, moreover, let X^*_i be the path from x_i to w_2 on the outer walk of $H'' - w_1$ without going through x_{3-i}. And let A_0 be a path in G from a_0 to y_1 and internally disjoint from B'.

Suppose H contains disjoint paths from y_1, y_2 to x_2, x'_1, respectively, and internally disjoint from A, as well as disjoint paths from y_1, y_2 to x_1, x'_2, respectively, and internally disjoint from A. Then, by Lemma 2.1.7, for any $i \in [2]$, H'' does not contain disjoint paths from w_1, w_2 to x_i, x''_{3-i}, respectively, and internally disjoint from A. This contradicts (iii) of Lemma 2.3.2.

Hence, by symmetry, we may assume that H contains no disjoint paths from y_1, y_2 to x_1, x'_2, respectively, and internally disjoint from A. Then by Lemma 2.3.2, H contains disjoint paths Y'_1, Y'_2 from y_1, y_2 to x_2, x'_1, respectively, and internally disjoint from A.

Then by Lemma 2.1.7 and 2.3.2, we may further assume H'' contains disjoint paths Y_1'', Y_2'' from w_1, w_2 to x_2, x_1'', respectively, and internally disjoint from A, but no disjoint paths from w_1, w_2 to x_1, x_2'', respectively, and internally disjoint from A. Moreover, by (i) of Lemma 2.3.2, $H - \{y_1, y_2\} \cup V(A(x_1, x_2))$ contains a path D' from x_1 to x_2, and $H'' - \{w_1, w_2\} \cup V(A(x_1, x_2))$ contains a path D'' from x_1 to x_2.

(1) There is no A-B' path in G from $A(x_1, x_2)$ to $B'(w_1, y_1)$.

For, suppose that P is an A-B' path from $p \in V(A(x_1, x_2))$ to $p' \in V(B'(w_1, y_1))$. Then $G_0' - B'(w_2, w_1) - B'[y_2, b_2]$ does not contain disjoint paths B_1^*, A_0^* from b_1, a_0 to p', y_1, respectively; otherwise, $A[a_1, x_1] \cup D'' \cup A[x_2, a_2] \cup Y_1' \cup A_0^*$ and $B_1^* \cup P \cup A(x_1, x_2) \cup Y_2' \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. Hence, there exists $w' \in V(B'(w_2, w_1))$, $a_0' \in V(G_0')$, and $b_2' \in V(B'[y_2, b_2])$, such that $\{w', a_0', b_2'\}$ is a 3-cut in G_0' separating $B'[w', b_2']$ from $\{a_0, b_1, b_2\}$.

Now $b_1 = w_2$. For, suppose not. Since w_1, w_2 are feet of H'', w_1, w_2 are incident with some finite face of G_0'. Therefore, $\{w_2, a_0', b_2'\}$ is a 3-cut in G_0' separating $B'[w_2, b_2']$ from $\{a_0, b_1, b_2\}$, a contradiction to Lemma 2.3.7. Similarly, by the symmetry between H and H'', we can also prove $b_2 = y_2$.

Now, since $b_2' \in V(B'[y_2, b_2])$, $b_2' = b_2$. So $a_0' = a_0$; or else, $\{b_1, a_0', b_2\}$ is a 3-cut in G_0' separating a_0 from $B'(b_1, b_2)$, a contradiction. Then a_0, b_1, w', w_1 are incident with some finite face of G_0'. Similarly, by the symmetry between H and H'', a_0, b_2, y_1 are incident with some finite face of G_0', which implies $\alpha(A, B') = 0$.

By Lemma 2.3.2, $V(X_2'' \cap X_1^*) - \{w_2\} = \emptyset$. Now $\alpha(A, B') \geq 1$ by Lemma 2.2.1 and the following paths: the path $A_0 \cup Y_1' \cup A[x_2, a_2]$ from a_0 to a_2, the path $X_2'' \cup A(x_1, x_2) \cup Y_2'$ from b_1 to b_2, and the path $A[a_1, x_1] \cup X_1^*$ from a_1 to b_1. This is a contradiction. $\Box$

(2) $a_1 = x_1$ and $a_2 = x_2$.

Recall that for $i \in [2]$, $P_{1,i}$ and $P_{2,3-i}$ are disjoint paths from x_1, x_2 to y_i, y_{3-i}, respectively, in $H - A(x_1, x_2)$. For $i \in [2]$, let $Q_{1,i}, Q_{2,3-i}$ be disjoint paths from x_1, x_2 to w_i, w_{3-i}, respectively, in $H'' - A(x_1, x_2)$.

We claim that for $i \in [2]$, G has no edge from $A[a_i, x_i)$ to $B'(b_1, w_2]$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A[a_i, x_i)$ and $b' \in B'(b_1, w_2]$. Then $b_1 \neq w_2$. By Lemma 2.3.7, $G_0' - B'[b', w_2] - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to w_1, y_1, respectively. Now $A[a_i, a'] \cup e' \cup B'[b', w_2] \cup Q_{3-i,2} \cup A[x_{3-i}, a_{3-i}] \cup P_{3-i,1} \cup A_0^*$ and $B_1^* \cup Q_{i,1} \cup P_{i,2} \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

Due to the symmetry between H and H'', with the same argument above, we can show that for $i \in [2]$, G has no edge from $A[a_i, x_i)$ to $B'[y_2, b_2)$. Thus, (2) follows from Lemma 2.3.4 and the assumption that G^* is 6-connected. □

(3) $H'' - X_1^* \cup X_2^*$ contains a path Q'' from w_1 to $A(x_1, x_2)$; and $H - X_1 \cup X_2$ contains a path Q from y_1 to $A(x_1, x_2)$.

By the symmetry between H and H'', we only prove the existence of Q''. Suppose for a contradiction that Q'' does not exist.

We see that $(N_G(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2])$. For, otherwise, by (ii) of Lemma 2.3.2, there exists $v'' \in N_G(w_1) \cap V(H'')$, $c_1'' \in A(x_1, x_2'')$, and $c_2'' \in X_2''(x_2'', w_2)$, such that $v'' \notin X_2'' \cup A(x_1, x_2]$, $\{c_1'', c_2''\}$ is a cut in $H'' - \{w_1, x_2\}$ separating v'' from x_1, and there exists a path P_1'' from v'' to c_1'' in $H'' - w_1 - x_2$, which is internally disjoint from $X_2'' \cup A[x_1, x_2'']$. But then, $w_1v'' \cup P_1''$ is a path from w_1 to $A(x_1, x_2)$ in $H'' - X_1^* \cup X_2^*$, a contradiction.

Now, since Q'' does not exist, combined with $(N_G(w_1) \cap V(H'')) \subseteq V(X_2'' \cup A(x_1, x_2])$, we may further assume $(N_G(w_1) \cap V(H'')) \subseteq V(X_2^*)$, contradicting (iii) of Lemma 2.3.1. □

(4) $b_1 = w_2$ and $b_2 = y_2$.

By the symmetry between H and H'', we only show $b_1 = w_2$. Suppose for a contradiction that $b_1 \neq w_2$.

Since w_1, w_2 are incident with some finite face of G_0', it follows from Lemma 2.3.7 that $G_0' - B'[w_2, w_1) - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to w_1, y_1, respectively.

Now, $A[a_1, x_1] \cup X_1^* \cup X_2^* \cup A[x_2, a_2] \cup Y_1' \cup A_0^*$ and $B_1^* \cup Q'' \cup A(x_1, x_2) \cup Y_2' \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

Note that G has no A-B' path from a_1 to $B'(w_1, y_1)$, as such a path together with Y_2'', Y_1', Y_2' forms a double cross, contradicting Lemma 2.1.7. So by (1) and (4), $\{b_1, b_2, w_1, y_1, a_2\}$ is a cut in G separating a_0 from a_1, a contradiction. □

We now use A, B' to form a new frame A', B', called *core* frame.

Lemma 2.3.9 *Let M_0 denote the union of all the A-B' bridges that are disjoint from $H - A - y_1$. Then there exists an induced path $A' \subseteq (A \cup M_0) - B'$ from a_1 to a_2 in G, such that $A'[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$ and the following hold:*

(i) A', B' is a good frame in γ.

(ii) Each A'-B' bridge lying on $B'[r_1, y_1]$ is contained in some A-B' bridge.

(iii) There exists an induced subgraph H^* in G, such that $A'[x_1, x_2] \cup H \subseteq H^*$, all A'-B' bridges not lying on $B'[r_1, y_1]$ are contained in H^*, and H^* is separated from $\{a_0, a_1, a_2, b_1, b_2\}$ by $V(A'[x_1, x_2]) \cup \{y_1, y_2\}$ in G.

(iv) For any $v \in (V(H^*) - V(A') \cup \{y_1\})$, $H^* - y_1$ contains a path from v to y_2 and internally disjoint from A'.

(v) If l, r are the extreme hands of an A'-B' bridge lying on $B'[r_1, y_1]$ then $\{l, r\} \neq \{x_1, x_2\}$, and $H^* - y_1$ does not contain a path from y_2 to $A'(l, r)$ and internally disjoint from A'.

Proof. We choose the induced path A' so that $A' \subseteq A \cup M_0 - B'$ is from a_1 to a_2, such that $A'[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$, (i)-(iv) are satisfied, and, subject to this, H is maximal. Note that such A' exists, as A satisfies (i)-(iv).

To prove (v), let M be an A'-B' bridge M lying on $B'[r_1, y_1]$ with extreme hands l, r and feet l', r'. If $\{l, r\} = \{x_1, x_2\}$ then, since M is contained in an A-B' bridge (by (ii)), M is contained in a main A-B' core, a contradiction to Lemma 2.3.8. Hence, $H - y_1$ contains a path Y_2 from y_2 to $y_2' \in A'(l, r)$ and internally disjoint from A'.

Let T be an induced path in $M - A'(l, r) \cup B'[l', r']$ from l to r, and let $C_1, C_2, \ldots, C_n$ be the components of $M \cup A'[l, r] \cup B'[l', r'] - T$ not containing $A'(l, r)$ and not containing $B'[l', r']$. We choose T, such that $|T| := (-|V(\bigcup_{i \in [n]} C_i)|, |V(C_1)|, |V(C_2)|, \ldots, |V(C_n)|)$ is maximal with respect to the lexicographical ordering.

We claim $n = 0$. For, suppose $n > 0$. Let $l_n, r_n \in N_G(C_n) \cap V(T)$ such that $T[l_n, r_n]$ is maximal. Since G^* is 6-connected, there exists another component C of $M \cup A'[l, r] \cup B'[l', r'] - T$, such that $N_G(C) \cap T(l_n, r_n) \neq \emptyset$. Now, let T' be an induced path in $G[T \cup C_n]$ between l_n and r_n, such that $T' \cap T(l_n, r_n) = \emptyset$. Clearly, $|T'| > |T|$, a contradiction.

Now, let A'' be obtained from A' by replacing $A'[l, r]$ with T. Clearly, $A''[a_i, x_i] = A[a_i, x_i]$ for $i \in [2]$. Since T is induced, A'' is induced. Moreover, since $n = 0$, then any component of $G[V(M \cup A'[l, r] \cup B'[l', r'])] - T$ contains $A'(l, r)$ or $B'[l', r']$, and so $G - A''$ is connected. Hence, A'', B' is a frame. Since $A_0''(B') = A_0'(B') = A_0(B')$, we see that A'', B' is a good frame in γ.

Next, we show that G has no A'-B' path from $A'(l, r)$ to $B'[b_1, y_1)$ and disjoint from T. For otherwise, let S be an A'-B' path from $s \in A'(l, r)$ to $s' \in B'[b_1, y_1)$ and disjoint from T. Then A'' and $B'[b_1, s'] \cup S \cup A'[s, y_2'] \cup Y_2 \cup B'[y_2, b_2]$ are disjoint paths from a_1, b_1 to a_2, b_2, respectively, in $G - (A_0(B') - B') - y_1$, a contradiction to (i) of Lemma 2.2.2.

Hence, there does not exist an A'-B' bridge N lying on $B'[r_1, y_1]$, such that $N \neq M$, $N \cap A'(l, r) \neq \emptyset$, and $N \cap B'[b_1, y_1) \neq \emptyset$. So each A''-B' bridge lying on $B'[r_1, y_1]$ must be contained in some A'-B' bridge and, hence, contained in some A-B' bridge. So A'', B' satisfies (ii).

And $V(A''[x_1, x_2]) \cup \{y_1, y_2\}$ is a cut in G separating $V(H)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Now, we let V'' be the set of vertices of $A'' \cup B'[b_1, y_1] \cup B'[y_2, b_2]$-bridge of G containing

$A'(l,r)$, and let $H'' := G[V'' \cup V(A''[x_1,x_2])]$. Then clearly (iii) and (iv) holds for A'', B'. However, H'' properly contains H, a contradiction. □

2.4 Inside the main A'-B' core

We use the notation of the previous section, in particular, Lemma 2.3.3 and 2.3.9: γ is infeasible, A', B' is a core frame, and let $H' := H^* - \{x_1y_2, x_2y_2\}$, where B', t_1, t_2, R_1, r_1 are defined as in or after Lemma 2.3.3, $A', H^*, x_1, x_2, y_1, y_2$ are defined as in Lemma 2.3.9. We also say that H' is the main A'-B' core in γ with extreme hands x_1, x_2 and feet y_1, y_2 (such that y_2 is the main foot).

We now study the structure of G inside H'.

Lemma 2.4.1 *$(H' - y_1, A'[x_1,x_2], y_2)$ is planar, the degree of y_2 in $H' - y_1$ is at least 2, and $H' - y_1 - A'(x_1,x_2)$ contains disjoint paths from y_1, y_2 to x_i, x_{3-i}, respectively, for $i \in [2]$. Moreover, for $i \in [2]$, let X_i be the path from x_i to y_2 on the outer walk of $H' - y_1$ without going through x_{3-i}, then $N_G(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i \in [2]$.*

Proof. We can apply the same proof in Lemma 2.2.4, and show that $(H'-y_1, A'[x_1,x_2], y_2)$ is planar, and $N_G(y_1) \cap V(H' - y_1 - A') \not\subseteq V(X_i)$ for $i = 1, 2$.

Moreover, since $V(H - y_1) \subseteq V(H' - y_1)$, then, by (iii) of Lemma 2.3.1, the degree of y_2 in $H' - y_1$ is at least 2, and $H' - A'(x_1,x_2) - \{y_1x_1, y_1x_2\}$ contains disjoint paths from y_1, y_2 to x_1, x_2, respectively, as well as disjoint paths from y_1, y_2 to x_2, x_1, respectively. □

Lemma 2.4.2 *Let R be an A'-B' path from $r \in V(A'(x_1,x_2))$ to $r' \in V(B'[r_1,y_1])$ such that $B'[r_1,r']$ is minimal. If $r' \neq r_1$ then the following conclusions hold:*

(i) There exists an A-B core H_1 with r_1 as a foot.

(ii) Let r_2 be the other foot of H_1, then there exists an A'-B' bridge with r_1 as a foot, intersecting A' only at x_j for some $j \in [2]$, and lying on $B'[r_1,r_2]$.

(iii) $r' \in B'(r_1, r_2)$*, and* G *has an* A'*-*B' *bridge with feet* l'_1, r'_1*, which is internally disjoint from* R *and intersecting* A' *only at* x_j*, such that* $r' \in B'(l'_1, r'_1)$*.*

(iv) If G'_0 *has a cut* $\{a'_0, b'_1, b'_2\}$ *separating* $B'[b'_1, b'_2]$ *from* $\{a_0, b_1, b_2\}$ *such that* $b'_1 \in V(B'(r_1, r'])$ *and* $b'_2 \in V(B'[y_2, b_2])$*, then* $r_1 = b_1$*,* $a'_0 = a_0$*,* G'_0 *has no path from* a_0 *to* b_1 *and internally disjoint from* B'*, and* $\alpha(A', B') \le 1$*.*

Proof. To prove (i), assume that r_1 is not a foot of any A-B core. Then by the definition of r_1, G has an edge from r_1 to $a' \in V(A(x_1, x_2))$. Since $r' \neq r_1$, $a' \notin A'(x_1, x_2)$. Moreover, a' is not contained in any A'-B' bridge lying on $B'[r_1, y_1]$, as any such A'-B' bridge is contained in an A-B' bridge (by (ii) of Lemma 2.3.9). So $a' \in V(H' - y_1) \setminus V(A')$. Hence, by (iv) of Lemma 2.3.9, $H' - y_1$ has a path Y_2 from a' to y_2 and internally disjoint from A'. Therefore, A' and $B'[b_1, r_1] \cup r_1 a' \cup Y_2 \cup B'[y_2, b_2]$ are disjoint paths from a_1, b_1 to a_2, b_2, respectively, in $G - V(A'_0(B') - B') \cup \{y_1\}$, contradicting (i) of Lemma 2.2.2.

Now, we prove (ii). By Lemma 2.3.4, r_2 is the main foot of H_1. Hence, by (iii) of Lemma 2.3.1, r_1 has two neighbors u_1, u_2 in $H_1 - r_2 - A$. Since $B'[r_1, r_2]$ is induced in $G - \{r_1 r_2\}$ (by Lemma 2.3.3), $u_p \notin B'$ for some $p \in [2]$. Moreover, $u_p \notin A'(x_1, x_2)$ since $r' \neq r_1$. Thus, u_p must be contained in some A'-B' bridge M_0 lying on $B'[r_1, r_2]$, which must have r_1 as a foot and cannot have both x_1 and x_2 as extreme hands (by (v) of Lemma 2.3.9). Hence, since $r' \neq r_1$, this A'-B' bridge intersect A' only at x_j for some $j \in [2]$.

Obviously, since G^* is 6-connected, $r' \in B'(r_1, r_2)$ to avoid the cut $\{r_1, r_2, x_1, x_2\}$ in G^* separating $V(H_1)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. Let l'_0, r'_0 be the feet of M_0 with $l'_0 = r_1$ and $r'_0 \in B'[r_1, r_2]$. For, suppose (iii) fails. Then $r' \in B'[r'_0, r_2]$. Since $x_{3-j} \notin V(H_1 \cap A')$ (by Lemma 2.3.8), then by the definition of r', $\{x_j, r_1, r'\}$ is a cut in G separating M_0 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

To prove (iv), we observe that $B'[r_1, r_2]$ is on the boundary of a finite face of G'_0. Therefore, since $r' \in B'(r_1, r_2)$, a'_0 and r_1 are also incident with that finite face. Suppose $r_1 \neq b_1$ or $a'_0 \neq a_0$. Then $\{a'_0, r_1, b'_2\}$ is a 3-cut in G'_0 separating $B'[r_1, b'_2]$ from $\{a_0, b_1, b_2\}$.

By Lemma 2.3.7, $r_1 = b_1$. So $a_0' \neq a_0$. Then, by Lemma 2.3.7, $\{a_0', b_1, b_2', b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction. So, $r_1 = b_1$ and $a_0' = a_0$. Hence, G_0' has no path that is from a_0 to b_1 and internally disjoint from B'. In particular, $\alpha(A', B') \leq 1$. $\square$

Since G^* is 6-connected, G has two disjoint A'-B' paths P, Q from $p, q \in V(A'(x_1, x_2))$ to $p', q' \in V(B'[r_1, y_1))$, respectively. We choose P, Q so that

(i) $A'[p, q]$ is maximal,

(ii) subject to (i), $B'[b_1, p'] \cap B'[b_1, q']$ is minimal, and

(iii) subject to (ii), $B'[p', q']$ is maximal.

By the symmetry between a_1 and a_2, we may relabel a_1, x_1, x_2, a_2 so that

- a_1, x_1, p, q, x_2, a_2 occur on A' in order, and $b_1, r_1, p', q', y_1, b_2$ occur on B' in order.

Lemma 2.4.3 *Any A'-B' path from $B'[r_1, p')$ to $A'(x_1, x_2)$ must be disjoint from P, Q, and end in $A'(p, q)$. Moreover, if $H' - y_1$ contains a path from $u \in A'[q, x_2)$ to y_2 and internally disjoint from A', then all A'-B' paths from $A'(u, x_2)$ to $B'[r_1, y_1]$ and internally disjoint from $H' - y_1$ are edges ending in $\{r', y_1\}$.*

Proof. First, assume S is an A'-B' path from $s' \in V(B'[r_1, p'))$ to $s \in V(A'(x_1, x_2))$. Then $V(S \cap (P \cup Q)) = \emptyset$; for otherwise, let $v \in V(S \cap (P \cup Q))$ with $S[s', v]$ minimal then $P' := S[s', v] \cup P[v, p]$ and Q (when $v \in V(P)$) or P and $Q' := S[s', v] \cup Q[v, q]$ (when $v \in V(Q)$) contradict the choice of P, Q. Hence, $s \in A'(p, q)$ as otherwise S, P or S, Q contradict the choice of P, Q.

Now let Y_2 be a path in $H' - y_1$ from $u \in V(A'[q, x_2))$ to y_2 and internally disjoint from A'. We first see that G has no path from $A'(u, x_2)$ to $B'[r_1, y_1) - p'$. For, suppose not. Let S be an A'-B' path from $s \in V(A'(u, x_2))$ to $s' \in V(B'[r_1, y_1) - p')$. Then $V(S \cap P) \neq \emptyset$, or else, P, S contradict the choice of P, Q. Since $s' \neq p'$, S, P are contained in an A'-B' bridge. However, by $u \in A'(p, s)$, the existence of Y_2 contradicts (v) of Lemma 2.3.9.

Now let S be an arbitrary A'-B' path from $s \in A'(u, x_2)$ to $s' \in B'[r_1, y_1]$. Suppose S has length at least 2. Then S is contained in some A'-B' bridge N with feet n'_1, n'_2 and extreme hands n_1, n_2. Then $n'_1, n'_2 \in \{p', y_1\}$. By (v) of Lemma 2.3.9 and the existence of S and Y_2, $A'[n_1, n_2] \subseteq A[u, x_2]$. Let $h_1, h_2 \in A'[x_1, x_2]$, such that $A'[n_1, n_2] \subseteq A'[h_1, h_2]$, $H' - y_1$ does not contain a path from $A'(h_1, h_2)$ to y_2 and internally disjoint from A', and subject to this, $A'[h_1, h_2]$ is maximal. Clearly, $A'(h_1, h_2) \subseteq A'(u, x_2)$, and for $i \in [2]$, $H' - y_1$ contains a path from h_i to y_2 and internally disjoint from A'. By (v) of Lemma 2.3.9, $\{h_1, h_2, p', y_1\}$ is a cut in G^* separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Thus, S must be an edge. To complete the proof, we need to show $r' = p'$. For, suppose $r' \neq p'$. By (i), R is disjoint from P, Q with $r \in A'(p, q)$, and so R, P, S, Y_2 force a double cross in A, B, contradicting Lemma 2.1.7. □

Let $R = P$ if $r' = p'$, and if $r' \neq p'$ then by Lemma 2.4.3, R is disjoint from P, Q with $r \in A'(p, q)$ (seen at Figure 2.8). By Lemma 2.4.1, for $i \in [2]$, we let $P_{1,i}, P_{2,3-i}$ be disjoint paths from x_1, x_2 to y_i, y_{3-i}, respectively, in $H' - y_1 - A'(x_1, x_2)$.

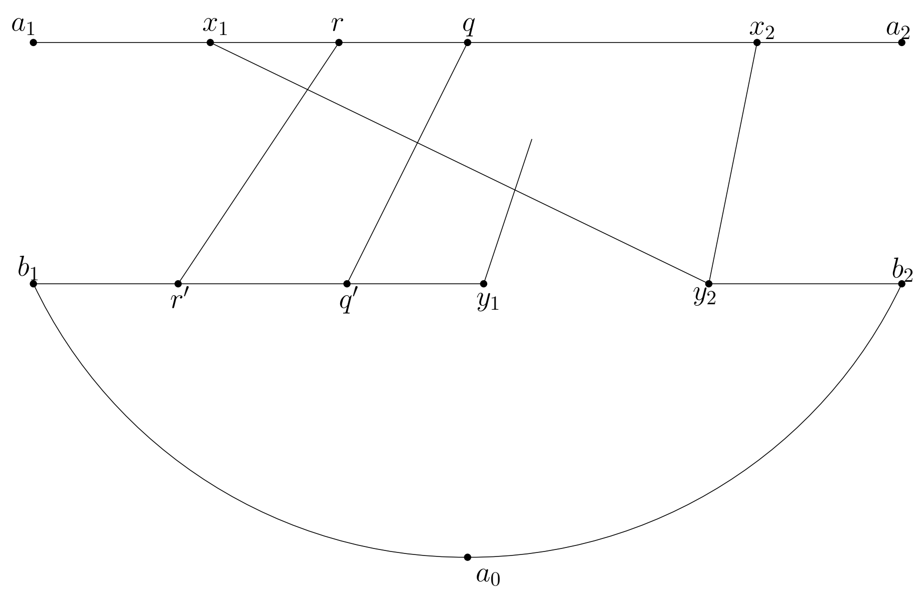

Figure 2.8: A core frame

We now use the structure inside H' to derive further structure outside H'.

Lemma 2.4.4 *(i) G has no edge from $B'(b_1, r_1]$ to $A'(x_2, a_2]$ and no edge from $B'[y_2, b_2)$ to $A'[a_1, x_1)$.*

(ii) G has no edge from b_1 to $A'[a_1, x_1] \cup A'[x_2, a_2]$ and no edge from b_2 to $A'[x_2, a_2]$.

(iii) $r_1 = b_1$ implies $x_1 = a_1$, and $y_2 = b_2$ implies $x_2 = a_2$.

(iv) If $y_2 \neq b_2$ and y_2 is a cut vertex of G'_0 separating b_2 from $\{a_0, b_1\}$, then $N_G(b_2) = \{y_2, x_1\}$, $a_1 \neq x_1$, and $a_2 = x_2$.

Proof. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, we may assume that

(1) when $b_1 \neq r_1$, $G'_0 - B'(b_1, r'] - B'[y_2, b_2]$ contains disjoint paths B_1^*, A_0^* from b_1, a_0 to q', y_1, respectively.

(2) G has no edge from $A'(x_2, a_2]$ to $B'(b_1, r_1]$.

For, let $e = ab \in E(G)$ with $a \in A'(x_2, a_2]$ and $b \in B'(b_1, r_1]$. Then $b_1 \neq r_1$; so B_1^*, A_0^* exist by (1). Now $A'[a_1, r] \cup R \cup B'[b, r'] \cup e \cup A'[a, a_2] \cup P_{1,1} \cup A_0^*$ and $B_1^* \cup Q \cup A'[q, x_2] \cup P_{2,2} \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. □

(3) G has no edge from b_2 to $A'[x_2, a_2]$.

For, let $e = ab_2 \in E(G)$ with $a \in A'[x_2, a_2]$. Then $a \neq a_2$ and let $e' = a_2 b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'[y_2, b_2)$ to avoid the double cross $e, e', P_{1,2}, P_{2,1}$. Hence, $b' \in B'(b_1, r_1]$, contradicting (2). □

(4) G has no edge from $A'[a_1, x_1)$ to $B'[y_2, b_2)$.

Otherwise, let $e = ab \in E(G)$ with $a \in A'[a_1, x_1)$ and $b \in B'[y_2, b_2)$. Then G has no edge from b_2 to $\{x_1, x_2\}$; as such an edge must be $b_2 x_1$ by (3), which forms a double cross with e, $P_{1,1}$ and $P_{2,2}$, contradicting Lemma 2.1.7.

Hence, by Lemma 2.3.7 and (iv) of Lemma 2.4.2, $G'_0 - B'[b_1, r'] - B'[y_2, b]$ has disjoint paths B_2, A_0 from b_2, a_0 to y_1, q', respectively. But then, $A'[a_1, a] \cup e \cup B'[y_2, b] \cup P_{2,2} \cup$

$A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[x_1, r] \cup P_{1,1} \cup B_2$ show that γ is feasible, a contradiction. $\square$

(5) (i) and (ii) hold.

For, suppose not. Then G has an edge $e = b_1 a$ with $a \in A'[a_1, x_1] \cup A'[x_2, a_2]$.

Suppose $a \in A'[a_1, x_1]$. Then $a \neq a_1$, and let $e' = a_1 b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,2}, P_{2,1}$. So $b' \in B'[y_2, b_2)$, contradicting (4).

Hence, $a \in A'[x_2, a_2]$. Then $a \neq a_2$, and let $e' = a_2 b' \in E(G)$ with $b' \in B'(b_1, b_2)$. Now $b' \notin B'(b_1, r_1]$ to avoid the double cross $e, e', P_{1,1}, P_{2,2}$. Hence, $b' \in B'[y_2, b_2)$.

If G has an edge e_3 from b_2 to $\{x_1, x_2\}$ then, by (3), it ends with x_1. So $a_1 \neq x_1$, and G has an edge e_4 from a_1 to $B'(b_1, b_2)$. But now, e, e', e_3, e_4 force a double cross, a contradiction.

So G has no edge from b_2 to $\{x_1, x_2\}$. Hence, by Lemma 2.3.7, $G_0' - B'[b_1, r_1] - B'[y_2, b']$ has disjoint paths B_2, A_0 from b_2, a_0 to y_1, q', respectively. But then, $A'[a_1, q] \cup P_{1,2} \cup B'[y_2, b'] \cup e' \cup Q \cup A_0$ and $e \cup A'[x_2, a] \cup P_{2,1} \cup B_2$ show that γ is feasible, a contradiction. $\square$

Since G^* is 6-connected, it follows from (2) and (4) that (iii) holds. It remains to prove (iv). So assume $y_2 \neq b_2$ and y_2 is a cut vertex of G_0' separating b_2 form $\{a_0, b_1\}$. Then $\alpha(A', B') \leq 1$.

Suppose $B'(y_2, b_2) \neq \emptyset$. Then, since G^* is 6-connected, it follows from (4) that G has edges from $B'(y_2, b_2)$ to distinct $u_1, u_2 \in V(A'[x_2, a_2])$, and we choose u_1, u_2 so that $A'[u_1, u_2]$ is maximal. Now, by (2) and (3), $\{u_1, u_2, y_2, b_2, x_1\}$ is a cut in G^* separating $B'(y_2, b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

So $B'(y_2, b_2) = \emptyset$. Then $a_2 = x_2$; for otherwise, since G^* is 6-connected, G has an edge from a_2 to $B'(b_1, r_1]$, contradicting (2). We may assume that there exists $e = b_2 a \in E(G)$ with $a \in A'(a_1, x_1)$; as otherwise, (iv) holds. Let $e' = a_1 b' \in E(G)$ with $b' \in B'(b_1, b_2)$.

Then $b' \in B'(b_1, r_1]$ by (4); so $b_1 \neq r_1$, and B_1^*, A_0^* exist by (1). Now, by Lemma 2.2.1, we derive $\alpha(A', B') = 2$ with the following paths: the path $e' \cup B'[b_1, b']$ from a_1 to b_1, the path $A'[a_1, a] \cup e$ from a_1 to b_2, the path $B_1^* \cup Q \cup A'[x_1, q] \cup P_{1,2} \cup B'[y_2, b_2]$ from b_1 to b_2, and the path $A_0^* \cup P_{2,1}$ from a_0 to a_2. This contradicts $\alpha(A', B') \leq 1$ as A', B' is a good frame. □

Let H_0 denote the minimal union of blocks of $H' - y_1 - A'[q, x_2]$ containing X_1, let W denote the path between x_1 and y_2, such that W is contained in the outer walk of H_0, and for any vertex $v \in V(W - A')$, there exists a vertex $u \in V(A'[q, x_2])$, such that u, v are incident with a finite face of $H' - y_1$, and let $w_1 \in V(A' \cap W)$ with $A'[x_1, w_1]$ maximal. We further study the structure inside H'.

Lemma 2.4.5 *(i) $H_0 = H' - y_1 - A(w_1, x_2]$, and each vertex in $W(w_1, y_2]$ has at most two neighbors on $A'[q, x_2]$, inducing a subpath of A' with at most two vertices.*

(ii) $H' - \{y_1, y_2\} - A'(x_1, x_2)$ contains a path from x_1 to x_2.

Proof. Suppose (i) is not true. Then $H' - y_1$ has a $(H_0 \cup A'[q, x_2])$-bridge J which has exactly one vertex in $W(w_1, y_2]$ (by definition of H_0 and since $G - A'$ is connected) or some vertex $w \in V(W(w_1, y_2])$ has two neighbors on $A'[q, x_2]$ such that the subpath of A' between them has at least three vertices. In the first case, let $w \in V(J \cap H_0)$ and $u, v \in V(J \cap A')$ such that $J \cap A' \subseteq A'[u, v]$; and in the second case, let u, v be the neighbors of w on $A'[q, x_2]$ such that $A'[u, v]$ is maximal. Then by Lemma 2.4.3, $\{u, v, w, y_1, r'\}$ is a cut in G^*, a contradiction.

Now suppose (ii) is not true. Then there exists $v_0 \in V(A'(x_1, x_2))$ such that y_2, v_0 are incident with a finite face of $H' - y_1$. We further choose v_0 so that $A'[v_0, x_2]$ is minimal, and let (L_1, L_2) be a separation in $H' - y_1$ such that $V(L_1 \cap L_2) = \{y_2, v_0\}$, $x_1 \in V(L_1)$, and $x_2 \in V(L_2)$.

By Lemma 2.4.1, for each $j \in [2]$, $H' - A'(x_1, x_2)$ contains disjoint paths from y_1, y_2 to x_j, x_{3-j}, respectively. So for $j \in [2]$, $G[V(L_j) \cup \{y_1\}] - y_2$ contains a path T_j from y_1

to x_j and internally disjoint from A'.

We see that y_2, v_0 are not incident with some finite face of H_0. For otherwise, $v_0 \in A'(x_1, w_1]$, $x_1 \neq w_1$, and $W[w_1, y_2] \subseteq L_2$. Hence, T_1, $W[w_1, y_2]$, P and Q are disjoint, which form a double cross, a contradiction to Lemma 2.1.7.

Now, by the minimality of $A'[v_0, x_2]$ and planarity of $H' - y_1$, $v_0 \in A'[q, x_2)$. Therefore, by Lemma 2.4.3, $\{v_0, x_2, r', y_1, y_2\}$ is a cut in G^* separating $V(L_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

Lemma 2.4.6 *$w_1 \neq x_1$, and H_0 is 2-connected.*

Proof. Suppose this is false. Let $z \in V(H_0)$ such that $z = x_1$ (when $x_1 = w_1$) or z is a cut vertex of H_0 and, subject to this, $W[x_1, z]$ is maximal. Then $V(W[z, y_2] \cap X_1) = \{z, y_2\}$. Note that $z \in X_1[x_1, y_2)$.

Let $w \in W(z, y_2]$ and $u \in N_G(w) \cap V(A'[q, x_2])$ such that $A'[u, x_2] \cup W[w, y_2]$ is maximal. Moreover, let K denote the $\{z, u\}$-bridge of $H' - y_1$ containing $A'[u, x_2] \cup X_2$, and let $K^* := G[V(K) \cup \{y_1\}]$.

By (v) of Lemma 2.3.9 and by the existence of $W[y_2, w] \cup wu$,

(1) no A'-B' bridge outside H' has one extreme hand in $A'[x_1, u)$ and the other in $A'(u, x_2]$.

Thus, since $\{y_1, y_2, z, u, x_2\}$ is not a cut in G^* separating K from $\{a_0, a_1, a_2, b_1, b_2\}$, G has an A'-B' path from $A'(u, x_2)$ to $B'[r_1, y_1)$ and internally disjoint from H'. By Lemma 2.4.3,

(2) all A'-B' paths from $A'(u, x_2)$ to $B'[r_1, y_1]$ and internally disjoint from H' are edges from $A'(u, x_2)$ to $\{r', y_1\}$.

So let $e = ar' \in E(G)$ with $a \in A'[u, x_2)$, and choose a such that $A'[u, a]$ is minimal. Let L denote the path on the outer walk of K between y_2 and u not going through x_2, and let $L_0 := L \cup A'[u, a]$. Then

(3) $V(L_0 \cap X_2) = \{y_2\}$ and $N_G(y_1) \cap V(K) \subseteq V(L_0)$.

First, suppose there exists $v \in V(L_0 \cap X_2)$, such that $v \neq y_2$. Then $\{v, y_1, u, x_2, r'\}$ is a cut in G^* separating $V(A'(u, x_2))$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Now suppose there exist $v \in N_G(y_1) \cap V(K)$ such that $v \notin V(L_0)$. We claim that $K^* - L_0$ has a path Y_1 from y_1 to x_2. For otherwise, by the planar structure of K, there exist $c_1, c_2 \in V(L_0)$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating v from x_2. Thus, by (2) and the choice of a, $\{c_1, c_2, y_1, u, z\}$ is a cut in G^* separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

If G has an A'-B' path T from $A'(x_1, u)$ to $B'(r', y_1]$ and internally disjoint from H', then T, e, L, Y_1 force a double cross, a contradiction. So T does not exist. Then $u = q$ and, by (1), $\{x_1, u, z, r'\}$ is a cut in G^* separating r from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

We will need the following claim.

(4) G'_0 contains a path A^*_0 from $B'(r', y_1)$ to a_0 and internally disjoint from B'.

For otherwise, there exists $b'_1 \in V(B'[b_1, r'])$, such that $\{b'_1, y_1\}$ is a 2-cut in G'_0 separating $B'[b'_1, y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b'_1, y_1, y_2\}$ is a 3-cut in G'_0 separating $B'[b'_1, y_2]$ from $\{a_0, b_1, b_2\}$. We choose b'_1 so that $B'[b_1, b'_1]$ is minimal. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, $b'_1 = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in G^* separating a_0 from $\{a_1, a_2\}$, a contradiction. $\square$

Let $y'_1, y''_1 \in V(L_0) \cap N_G(y_1)$ such that a, y'_1, y''_1, y_2 occur on L_0 in order and, subject to this, $L_0[y'_1, y''_1]$ is maximal.

(5) $y''_1 \in L_0[z, u)$.

For, otherwise, $y''_1 \in L_0(z, y_2]$. Then $y'_1 \notin L_0[z, y_2]$; otherwise, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, z, y_1, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, z, y_1, y_2, x_2)$ is planar, which contradicts Lemma 2.1.3.

We claim that $K - L_0[y_1', a] \cup L_0[y_2, y_1'']$ contains a path X' from x_2 to z. For otherwise, by (3) and the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating x_2 from z. If $c_1 \in A'[u, a]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in G^* separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_1 \notin A'[u, a]$. Then G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, c_1, c_2, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[u, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', u, c_1, c_2, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Now, the following paths give a contradiction to (i) of Lemma 2.2.2: the path $A'[a_1, x_1] \cup X_1[x_1, z] \cup X' \cup A'[x_2, a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0. $\square$

Now $y_1' \in A'(u, a]$. For, otherwise, $y_1', y_2'' \in L_0[z, u]$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u, y_1, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $G_2 = K^*$, and $(G_2, r', u, y_1, z, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Moreover, $K - L_0[y_1', a] \cup L_0[y_2, y_1'']$ contains a path X' from x_2 to u. For otherwise, by (3) and the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$ and $c_2 \in V(L_0[y_2, y_1''])$, such that c_1, c_2 are incident with a finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating x_2 from u. If $c_2 \in L_0[y_2, z]$ then $\{c_1, c_2, y_2, x_2, r'\}$ is a cut in G^* separating $V(X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. So $c_2 \notin L_0[y_2, z]$. Then G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', c_1, c_2, z, y_2, x_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, x_2] \cup X_2) \subseteq V(G_2)$, and $(G_2, r', c_1, c_2, z, y_2, x_2)$ is planar. This contradicts Lemma 2.1.3.

Hence, the following paths contradict (i) of Lemma 2.2.2: the path $A'[a_1, u] \cup X' \cup A'[x_2, a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0. $\square$

Lemma 2.4.7 *Let $z_1, z_2 \in V(W)$ with $W[z_1, z_2]$ is maximal, such that x_1, z_1, z_2, y_2 occur on W in order, and for each $i \in [2]$, $G[H_0 + y_1]$ has a path Z_i from y_1 to z_i and internally disjoint from W. Then, $N_G(y_1) \cap V(X_1[x_1, y_2)) = \emptyset$ and $Z_1 \cap (X_1 \cup X_2) = \emptyset$.*

Proof. By Lemma 2.4.6, $w_1 \neq x_1$ and H_0 is 2-connected. So $V(X_1 \cap W) = \{x_1, y_2\}$.

If $N_G(y_1) \cap V(X_1[x_1, y_2)) \neq \emptyset$ or $Z_1 \cap X_1 \neq \emptyset$ then $Z_1 \cup X_1$ contains a path S from y_1 to x_1 and disjoint from $W[w_1, y_2]$. Now S, $W[w_1, y_2]$, P, and Q force a double cross, contradicting Lemma 2.1.7. So $N_G(y_1) \cap V(X_1[x_1, y_2)) = \emptyset$ and $Z_1 \cap X_1 = \emptyset$.

Moreover, $Z_1 \cap X_2 = \emptyset$. For, otherwise, by the choice of z_1 and Z_1, it follows from the planarity of $H' - y_1$ that $z_1 \in V(X_2)$. But then, $H' - A'(x_1, x_2)$ contains no disjoint paths from y_1, y_2 to x_1, x_2, respectively. This contradicts Lemma 2.4.1. $\square$

Let $w_2, \cdots, w_m$ be the vertices on W in order from x_1 to y_2 such that for $i \in \{2, \cdots, m\}$, $N_G(w_i) \cap V(A'[q, x_2]) \neq \emptyset$.

Lemma 2.4.8 *$a_2 = x_2$, and if $y_2 \neq b_2$ then y_1, y_2 are cut vertices in G_0' separating b_2 from $\{a_0, b_1\}$, $N_G(b_2) = \{y_2, x_1\}$, and $a_1 \neq x_1$. Moreover, one of the following holds:*

(i) there exists a 2-cut $\{z_1', z_2'\}$ in H_0 with $x_1, z_1', z_1, z_2, z_2', y_2$ on W in order such that $W(z_1', z_2') \neq \emptyset$ and z_1', z_2' are incident with a finite face of H_0, or

(ii) $N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, y_2])$ and, for any $i \in [m]$, $w_i \notin W(z_1, z_2)$.

Proof. By Lemma 2.4.6, $w_1 \neq x_1$, and H_0 is 2-connected. If $y_2 = b_2$, then by (iii) of Lemma 2.4.4, we have $a_2 = x_2$.

Now assume $y_2 \neq b_2$. We claim that G_0' has a 3-cut $\{a_0', b_1', y_2\}$ with $b_1' \in B'[b_1, r_1]$, which separates $B'[b_1', y_2]$ from $\{a_0, b_1, b_2\}$. For otherwise, by (iv) of Lemma 2.4.2, $G_0' - B'[b_1, r'] - y_2$ contains disjoint paths A_0, B_2 from a_0, b_2 to q', y_1, respectively. Let Y_1 be a path in $Z_1 \cup W[z_1, w_1] \cup A'[w_1, r]$ from y_1 to r. Note that $r \notin A'[q, x_2]$ and, by Lemma 2.4.7, $Y_1 \cap (A'[q, x_2] \cup X_1 \cup X_2) = \emptyset$. Now, $A'[a_1, x_1] \cup X_1 \cup X_2 \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup Y_1 \cup B_2$ show that γ is feasible, a contradiction.

Thus, when $y_2 \neq b_2$, we may apply Lemma 2.3.7 (with $b_2' = y_2$), and conclude that $b_1' = b_1$, $a_0' = a_0$, and y_1, y_2 are cut vertices in G_0' separating b_2 from $\{a_0, b_1\}$. By (iv) of Lemma 2.4.4, we have $N_G(b_2) = \{y_2, x_1\}$, $a_1 \neq x_1$, and $a_2 = x_2$.

We now show (i) or (ii) holds. First, suppose $z_1 = z_2$. Then $N_G(y_1) \cap V(H_0) = \{z_1\}$; or else, there exists $v \in N_G(y_1) \cap V(H_0)$ with $v \neq z_1$, and $\{z_1, y_1\}$ is a cut in G separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Clearly, $z_1 \in V(W(w_1, y_2))$, and so (ii) holds.

So we may assume $z_1 \neq z_2$. Now suppose $W(z_1, z_2) \cap \{w_1, \ldots, w_m\} = \emptyset$. Then (ii) holds or there exists $v \in N_G(y_1) \cap V(H_0)$ such that $v \notin V(W)$. In the latter case, there exist $c_1, c_2 \in V(W(x_1, y_2])$, such that $\{c_1, c_2\}$ is a 2-cut in H_0 separating v from x_1; since, otherwise, $H_0 - W(x_1, y_2]$ contains a path T from v to x_1, and $y_1 v \cup T, W[w_1, y_2], R, Q$ force a double cross, contradicting Lemma 2.1.7. Now, $\{y_1, c_1, c_2\}$ is a cut in G^*, a contradiction.

Hence, we may assume $W(z_1, z_2) \cap \{w_1, \ldots, w_m\} \neq \emptyset$. Now suppose (i) fails. Then by the planar structure of H_0, $H_0 - W(x_1, z_1] - W[z_2, y_2]$ contains a path X' from x_1 to $W(z_1, z_2)$ and internally disjoint from W.

We claim that X' must be disjoint from Z_1, Z_2. For otherwise, let $x^* \in V(X' \cap Z_j)$ for some $j \in [2]$. As X', Z_1, Z_2 are all internally disjoint from W, $Z_j[s_j, x^*] \cup X'[x^*, x_1]$ implies that $z_1 = x_1$, contradicting Lemma 2.4.7 that $V(Z_1 \cap (X_1 \cup X_2)) = \emptyset$.

We claim $w_1 \in W(z_1, z_2)$. For otherwise, $w_i \in W(z_1, z_2)$ for some $i \geq 2$. Let $v_i \in N_G(w_i) \cap V(A'[q, x_2])$ with $A'[v_i, x_2]$ minimal. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, there exists a path A_0^* in G_0' from a_0 to $B'(r', y_1)$, which is internally disjoint from B'. Now $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup w_i v_i \cup A'[q, a_2] \cup Q \cup B'(r', y_1) \cup A_0^*$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

So $z_1 \in A'(x_1, w_1)$. Moreover, $r \notin A'(x_1, z_1]$; otherwise, $A'[a_1, x_1] \cup X' \cup W(z_1, z_2) \cup A'[w_1, a_2] \cup Q \cup B'(r', y_1) \cup A_0^*$ and $B'[b_1, r'] \cup R \cup A'[r, z_1] \cup Z_1 \cup Z_2 \cup W[z_2, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. But now, $A'[a_1, z_1] \cup Z_1 \cup B'(r', y_1] \cup A_0^* \cup Q \cup A'[q, a_2]$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. $\square$

Lemma 2.4.9 *Suppose (i) of Lemma 2.4.8 holds, and the 2-cut $\{z_1', z_2'\}$ in G_0' is chosen with $W[z_1', z_2']$ maximal. Then $z_1' \in A'[x_1, w_1]$ (seen at Figure 2.9).*

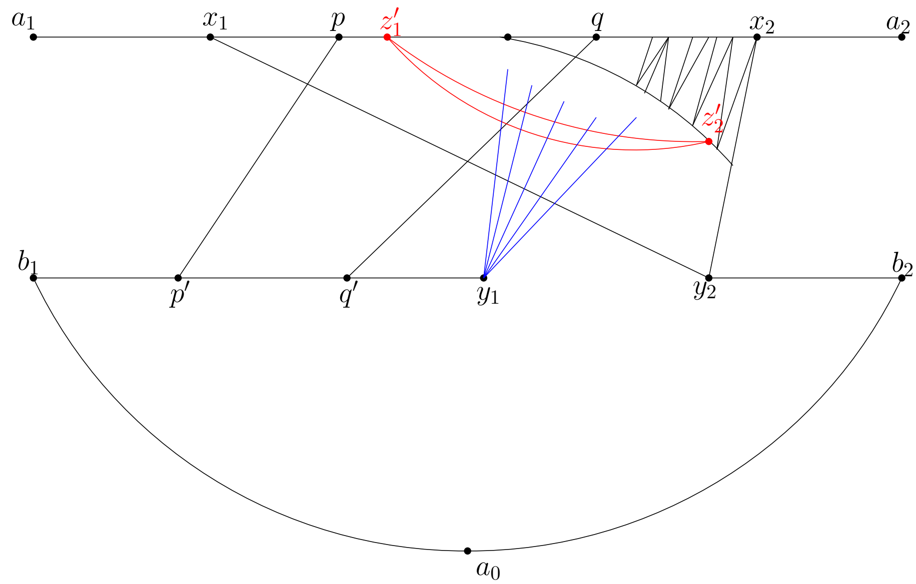

Figure 2.9: Structures in a core frame I

Proof. For, suppose $z_1' \notin A'[x_1, w_1]$. By Lemma 2.4.5, let $u', u'' \in V(A'[q, x_2])$ and $v', v'' \in V(W(z_1', z_2'))$ such that x_1, u', u'', x_2 occur on A' in order, $u'v', u''v'' \in E(G)$, and, subject to this, $A'[u', u'']$ is maximal and then $W[v', v'']$ is maximal. Then $H' - y_1$ has a separation (K, K') such that $V(K \cap K') = \{u', u'', z_1', z_2'\}$, $W[z_1', z_2'] \cup A'[u', u''] \subseteq K$, and $W[x_1, z_1'] \cup X_1 \subseteq K'$.

By (v) of Lemma 2.3.9 and by the existence of paths from y_2 to u', u'', respectively, in $H' - y_1$ that are internally disjoint from A',

(1) no A'-B' bridge outside H' has u' or u'' as internal vertex of the subpath of A' between its extreme hands.

Therefore, since $\{y_1, z_1', z_2', u', u''\}$ does not separate K from $\{a_0, a_1, a_2, b_1, b_2\}$ in G^*,

(2) $A'(u', u'') \neq \emptyset$, and G has an A'-B' path from $A'(u', u'')$ to $B'[r_1, y_1)$ and internally disjoint from $H' - y_1$.

Recall from Lemma 2.4.3 that

(3) all A'-B' paths from $A'(u', u'')$ to $B'[r_1, y_1]$ and internally disjoint from $H' - y_1$ are edges from $A'(u', u'')$ to $\{r', y_1\}$.

By (2) and (3), let $e = ar' \in E(G)$ with $a \in V(A'[u', u''))$ and $A'[u', a]$ minimal. Note that

(4) G_0' contains a path A_0^* from $B'(r', y_1)$ to a_0 and internally disjoint from B'.

For otherwise, there exists $b_1' \in B'[b_1, r']$, such that $\{b_1', y_1\}$ is a 2-cut in G_0' separating $B'[b_1', y_1]$ from $\{a_0, b_1, b_2\}$. Furthermore, $\{b_1', y_1, y_2\}$ is a 3-cut in G_0' separating $B'[b_1', y_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, $b_1' = b_1$, and $\{b_1, y_1, y_2, b_2\}$ is a cut in G separating a_0 from $\{a_1, a_2\}$, a contradiction. $\square$

Since $z_1' \notin A'[x_1, w_1]$, no finite face of K' incident with z_2' is incident with a vertex of $A'[x_1, w_1]$. Thus,

(5) $K' - A'[x_1, u']$ contains a path Y from y_2 to z_1' and internally disjoint from A'.

Let L denote the path on the outer walk of K from z_1' to u' without going through u'', and let $L_0 := L \cup A'[u', a]$. Note that $z_2' \notin V(L_0)$.

(6) $N_G(y_1) \cap V(K) \not\subseteq V(L_0) \cup \{z_2'\}$.

For, suppose $N_G(y_1) \cap V(K) \subseteq V(L_0) \cup \{z_2'\}$. Then $V(L_0) \cap N_G(y_1) \neq \emptyset$; otherwise, $\{u', u'', z_1', z_2', r'\}$ is a cut in G^* separating K from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Let $y_1', y_1'' \in V(L_0) \cap N_G(y_1)$, such that a, y_1', y_1'', z_1' occur on L_0 in order and $L_0[y_1', y_1'']$ is maximal.

We first claim $y_1' \in L_0(u', a]$. For otherwise, $y_1', y_2'' \in V(L_0[z_1', u'])$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', u', y_1, z_1', z_2', u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $(G_2, r', u', y_1, z_1', z_2', u'')$ is planar, contradicting Lemma 2.1.3.

Next, $y_1'' \in L_0[z_1', u')$. For, suppose $y_1'' \notin L_0[z_1', u')$. Then $y_1'' \in L_0[u', a]$. Now, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', y_1, u', z_1', z_2', u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(K) \subseteq V(G_2)$, and $(G_2, r', y_1, u', z_1', z_2', u'')$ is planar, contradicting Lemma 2.1.3.

We further claim $K - z_2' - L_0[z_1', y_1''] - L_0[y_1', a]$ contains a path X' from u'' to u'. For otherwise, by the planar structure of K, there exist $c_1 \in V(L_0[y_1', a])$, $c_2 \in V(L_0[z_1', y_1''])\,\cup$

$\{z_2'\}$, such that c_1, c_2 are incident with some finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating u' from u''. By the existence of the path $u''v'' \cup W[v'', v'] \cup v'u'$ from u'' to u', we may assume $c_2 = v'$. Moreover, $v' \neq v''$; otherwise, $\{v', u', u'', r', y_1\}$ is a cut in G^* separating $A'(u', u'')$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Now G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{r', c_1, v', z_1', z_2', u''\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(A'[c_1, u'']) \cup \{v''\} \subseteq V(G_2)$, and $(G_2, r', c_1, v', z_1', z_2', u'')$ is planar, which contradicts Lemma 2.1.3.

Now, the path $A'[a_1, u'] \cup X' \cup A'[u'', a_2]$ from a_1 to a_2, the path $B'[b_1, r'] \cup e \cup L_0[a, y_1'] \cup y_1'y_1 \cup y_1y_1'' \cup L_0[y_1'', z_1'] \cup Y \cup B'[y_2, b_2]$ from b_1 to b_2, and the path A_0^* from $B'(r', y_1)$ to a_0 contradict (i) of Lemma 2.2.2. $\square$

(7) $G[K + y_1] - V(L_0) \cup \{z_2'\}$ contains a path Y_1 from y_1 to u''.

Note that, by (6), there exists $v \in N_G(y_1) \cap V(K)$ such that $v \notin V(L_0) \cup \{z_2'\}$. So if (7) fails then, $K - z_2' - L_0$ has no path from v to u''; so there exist $c_1, c_2 \in V(L_0) \cup \{z_2''\}$, such that c_1, c_2 are incident with some finite face of K, and $\{c_1, c_2\}$ is a 2-cut in K separating v from u''. Thus, by (3) and the choice of a, $\{c_1, c_2, y_1, u', z_1'\}$ is a cut in G^* separating v from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

(8) $b_1 = r_1 = r'$, and G has no A'-B' path from $A'[a_1, u')$ to $B'(r', y_1]$ and internally disjoint from H'.

First, G has no A'-B' path from $A'[a_1, u')$ to $B'(r', y_1]$ and internally disjoint from H', to avoid forming a double cross with $e, Y \cup L, Y_1$.

Next we show $b_1 = r_1$ (and so $a_1 = x_1$ by (iii) of Lemma 2.4.4). For, suppose $b_1 \neq r_1$. By Lemma 2.3.7 and (iv) of Lemma 2.4.2, $G_0' - r' - B'[y_2, b_2]$ contains disjoint paths B_1, A_0 from b_1, a_0 to q', y_1, respectively. Now, $A'[a_1, r] \cup R \cup e \cup A'[a, a_2] \cup Y_1 \cup A_0$ and $B_1 \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction.

Moreover, $r_1 = r'$. For, suppose $r_1 \neq r'$. By (iii) of Lemma 2.4.2, there exists an A'-B' bridge M with feet l^*, r^*, such that M is internally disjoint from R, and $r' \in B'(l^*, r^*)$. Let

P^* be the path from l^* to r^* in M and internally disjoint from A', B', and let A'_0 be the path from a_0 to y_1 in G'_0 and internally disjoint from B'. Then $A'[a_1, r] \cup R \cup e \cup A'[a, a_2] \cup Y_1 \cup A'_0$ and $B'[b_1, l'_4] \cup P^* \cup B'[r'_4, q'] \cup Q \cup A'[q, u'] \cup L \cup Y \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. $\square$

Now, by (1), (3), (8), Lemma 2.4.3, and Lemma 2.4.8, $\{b_1, u', a_2, y_1, b_2\}$ is a cut in G^* separating a_0 from a_1, a contradiction. $\square$

Lemma 2.4.10 *y_1 is a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$, $\alpha(A', B') = 1$, and $G'_0 - B'(b_1, r'] - A'_0$ has a path B'_1 from b_1 to q', where A'_0 is the path from a_0 to y_1, which is in the outer walk of G'_0 and disjoint from $B' - y_1$.*

Proof. Recall the path Z_1 from Lemma 2.4.7. We claim that $H' - \{y_1, y_2\}$ contains a path X_0 from x_1 to x_2 and disjoint from $Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2)$. For otherwise, by the planar structure of $H' - y_1$, there exists a vertex $v \in V(Z_1 \cup W[z_1, w_1] \cup A'(x_1, x_2))$, such that y_2, v are incident with some finite face of H_0. By Lemma 2.4.5, $v \notin A'(x_1, x_2)$, and so $v \in V(Z_1 \cup W[z_1, w_1])$. If $v \in W[z_1, w_1]$ then (i) of Lemma 2.4.8 holds and the 2-cut $\{z'_1, z'_2\}$ can be chosen with $z'_2 = y_2$; so $z'_1 \in A'[x_1, w_1]$ by Lemma 2.4.9, contradicting Lemma 2.4.5. So $v \in Z_1 - z_1$, which implies that y_1 has a neighbor in $H_0 - W$; so (i) of Lemma 2.4.8 holds and the 2-cut $\{z'_1, z'_2\}$ still can be chosen with $z'_2 = y_2$. Again, $z'_1 \in A'[x_1, w_1]$ by Lemma 2.4.9, contradicting Lemma 2.4.5.

Now suppose y_1 is not a cut vertex in G'_0 separating b_2 from $\{a_0, b_1\}$. Then $y_2 = b_2$ by Lemma 2.4.8. If $G'_0 - B'[b_1, r'] - B'(y_1, b_2)$ contains disjoint paths A_0, B_2 from a_0, b_2 to q', y_1, respectively, then $A'[a_1, x_1] \cup X_0 \cup A'[q, a_2] \cup Q \cup A_0$ and $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, z_1] \cup Z_1 \cup B_2$ show that γ is feasible, a contradiction. Thus, such paths do not exist. Then by planarity, G'_0 has a 3-cut $\{a'_0, b'_1, b'_2\}$ with $b'_1 \in B'[b_1, r']$ and $b'_2 \in B'(y_1, b_2)$, which separates $B'(b'_1, b'_2)$ from $\{a_0, b_1, b_2\}$. Since y_1, b_2, b'_2 are incident with some finite face of G'_0, then a'_0, b_2 are incident with some finite face of G'_0, and so $\{b'_1, a'_0, b_2\}$ is a 3-cut in G'_0. Moreover, since y_1 is not a cut vertex in G'_0, then $a'_0 \neq a_0$. But now, by (iv) of Lemma 2.4.2, $b_1' \notin B'(r_1, r']$, and therefore, $b_1' \in B'[b_1, r_1]$. Now, by Lemma 2.3.7, $b_1' = b_1$. Then $\{b_1, b_2, a_0'\}$ is a cut in G^* separating a_0 from $\{a_1, a_2\}$, a contradiction.

Thus, y_1 is a cut vertex in G_0' and, hence, $\alpha(A', B') \leq 1$. Indeed, $\alpha(A', B') = 1$. To see this, let A_0' be the path from a_0 to y_1, which is in the outer walk of G_0' and disjoint from $B' - y_1$. When $y_2 = b_2$, let $B^* := A'[a_1, x_1] \cup X_1$; when $y_2 \neq b_2$, by Lemma 2.4.8, $x_1 b_2 \in E(G)$, and we let $B^* := A'[a_1, x_1] \cup x_1 b_2$. Then by Lemma 2.2.1, the following paths show $\alpha(A', B') = 1$: the path $A_0' \cup B'[q', y_1] \cup Q \cup A'[q, a_2]$ from a_0 to a_2, the path $B'[b_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2] \cup B'[y_2, b_2]$ from b_1 to b_2, and the path B^* from a_1 to b_2.

Finally, suppose $G_0' - B'(b_1, r'] - A_0'$ has no path B_1' from b_1 to q'. Then by planarity, G_0' has a 2-cut $\{a_0', b_1'\}$ with $a_0' \in V(A_0')$, $b_1' \in V(B'(b_1, r'])$, and a_0', b_1' cofacial, which separates b_1 from q'. Hence, $\{a_0', b_1', b_2\}$ is a 3-cut in G_0' separating $B'[b_1', b_2]$ from $\{a_0, b_1, b_2\}$. By Lemma 2.3.7, $b_1' \notin B'(b_1, r_1]$, and so $b_1' \in (r_1, r']$. But, by (iv) of Lemma 2.4.2, $r_1 = b_1$, $a_0' = a_0$, and G_0' has no path from a_0 to b_1 and internally disjoint from B'. Therefore, $\alpha(A', B') = 0$, a contradiction. $\square$

Lemma 2.4.11 *Suppose (i) of Lemma 2.4.8 does not hold and (ii) of Lemma 2.4.8 holds. Then $N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$ (seen at Figure 2.10).*

Proof. Note that in this case, $y_1 z_1, y_2 z_2 \in E(G)$. Since $z_1 \notin V(X_2)$ (by Lemma 2.4.7), $z_1 \notin W[w_m, y_2]$; so (ii) of Lemma 2.4.8 implies the existence of $j \in [m-1]$ with $z_1, z_2 \in W[w_j, w_{j+1}]$ and $z_2 \neq w_j$. We may assume $j \geq 2$ as otherwise the assertion holds. Thus, since (i) of Lemma 2.4.8 does not hold, $H_0 - W[x_1, w_1] - W[z_2, w_m]$ contains a path Y_2 from y_2 to w_2. Recall from Lemma 2.4.8 that $a_2 = x_2$, and recall paths B_1', A_0' from Lemma 2.4.10.

(1) $b_2 = y_2$.

For, suppose $b_2 \neq y_2$. Then by Lemma 2.4.8, G has an edge from b_2 to x_1, and $a_1 \neq x_1$. Let $a_1 b \in E(G)$ with $b \in V(B'(b_1, r_1])$. Now $\alpha(A', B') = 2$ by applying Lemma 2.2.1

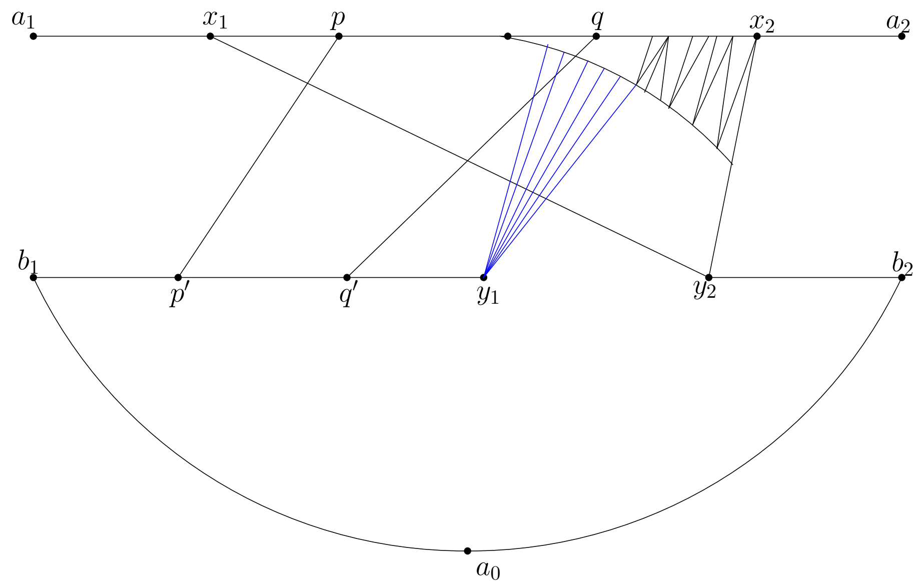

Figure 2.10: Structures in a core frame II

with the following paths: the path $A'_0 \cup y_1z_2 \cup W[z_2, w_m] \cup w_ma_2$ from a_0 to a_2, the path $B'_1 \cup Q \cup A'[w_1, q] \cup W[w_1, w_2] \cup Y_2 \cup B'[y_2, b_2]$ from b_1 to b_2, the path $a_1b \cup B'[b_1, b]$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup x_1b_2$ from a_1 to b_2 show that $\alpha(A', B') = 2$, contradicting Lemma 2.4.10. $\square$

Let $u_2 \in N_G(w_2) \cap V(A')$ with $A'[u_2, a_2]$ is maximal. Then

(2) $u_2 \neq x_2$.

For, suppose $u_2 = x_2$. Then G has an A'-B' path T from $t \in V(A'[a_1, w_1))$ to $t' \in V(B'[b_1, y_1])$ and internally disjoint from H'; as otherwise, $\{a_1, w_1, x_2, y_1, y_2\}$ is a cut in G^* separating H_0 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. We choose T so that $B'[b_1, t']$ is minimal and, subject to this, $A'[a_1, t]$ is minimal.

Then $t' \in B'[b_1, r']$ and G has no A'-B' path from $A'[a_1, t)$ to $B'[b_1, y_1]$ and internally disjoint from H'. For, if $t' \in B'(r', y_1]$ then, by the choice of T, we have $T \cap R = \emptyset$ and $r \in A'[w_1, q)$; now T, R, $y_1z_2 \cup W[z_2, w_m] \cup w_mx_2$, and $Y_2 \cup W[w_2, w_1]$ form a double cross, a contradiction. Now if G has an A'-B' path S from $s \in A'[a_1, t)$ to $s' \in B'[b_1, y_1]$ and internally disjoint from H', then by the choice of T, $T \cap S = \emptyset$ and $s \in B'(t', y_1]$; so T, S, $y_1z_2 \cup W[z_2, w_m] \cup w_mx_2$, and $Y_2 \cup W[w_2, w_1]$ form a double cross, a contradiction.

Now $V(T \cap Q) = \emptyset$. Otherwise, T, Q are contained in a same A'-B' bridge. Since $w_1 \in A'(t, q)$, the path from w_1 to y_2 in $H' - y_1$ contradicts (v) of Lemma 2.3.9.

Next, we show that $H_0 - (A'[x_1, t] \cup X_1[x_1, y_2) \cup W[z_1, w_j])$ contains a path Y_2' from y_2 to w_1. For otherwise, by the planar structure of H_0, there exist $c_1 \in V(W[z_1, w_j])$ and $c_2 \in V(A'[x_1, t]) \cup V(X_1[x_1, y_2))$, such that $\{c_1, c_2\}$ is a cut in H_0 separating y_2 from w_1. Recall that $j < m$ and $z_1 \notin V(X_2)$, and so $z_1 \in W[w_j, w_m)$. In fact, $c_2 \in A'(x_1, t]$; as otherwise $\{c_1, c_2, y_1, y_2, x_2\}$ is a cut in G^* separating w_m from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. Hence, $t \in A'(x_1, w_1)$. Since G has no A'-B' path from $A'[a_1, t)$ to $B'[b_1, y_1]$ and internally disjoint from H', G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{x_1, y_2, x_2, y_1, c_1, c_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, y_2, x_2, y_1, c_1, c_2)$ is planar, which contradicts Lemma 2.1.3.

Hence, by Lemma 2.2.1, the path $A_0' \cup z_1y_1 \cup W[z_1, w_j] \cup w_ja_2$ from a_0 to a_2, the path $B_1' \cup Q \cup A'[w_1, q] \cup Y_2'$ from b_1 to b_2, the path $A'[a_1, t] \cup T \cup B'[b_1, t']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2 show that $\alpha(A', B') = 2$, contradicting Lemma 2.4.10. $\square$

(3) G has no A'-B' path from $A'(u_2, a_2]$ to $B'(b_1, r']$.

For, suppose G has an A'-B' path S from $s \in A'(u_2, a_2]$ to $s' \in B'(b_1, r']$. Then, $A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup x_2w_m \cup W[w_m, z_2] \cup z_2y_1 \cup A_0'$ and $B_1' \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2$ show that γ is feasible, a contradiction. $\square$

(4) G has no disjoint A'-B' paths C, D from $c, d \in V(A'[x_1, x_2))$ to $c', d' \in V(B'[b_1, y_1])$ and internally disjoint from H', such that a_1, c, d, a_2 occur on A' in order, and b_1, d', c', y_1 occur on B' in order.

For, suppose such C, D exist. Then $c \notin A'[a_1, u_2)$; otherwise, C, D, $y_1z_2 \cup W[z_2, w_m] \cup w_mx_2$, and $Y_2 \cup w_2u_2$ form a double cross, a contradiction. So $d \in A'(u_2, x_2)$.

Then, by Lemma 2.4.3, $D = dd'$ and $d' = r'$. Moreover, by (3), $b_1 = r'$.

Now, G has no A'-B' path from $A'[a_1, u_2)$ to $B'(b_1, y_1]$ and internally disjoint from H'; otherwise, replace C by this path we have a contradiction to our claim that $c \notin A'[a_1, u_2)$. But then, by Lemma 2.4.3, $\{b_1, b_2, y_1, u_2, a_2\}$ is a cut in G^* separating a_1 from a_0, a contradiction. $\Box$

(5) $H_0 - A'(x_1, w_1] - W[z_2, y_2]$ has a path X' from x_1 to w_j.

For otherwise, by planarity of H_0, there exist $c_1 \in V(A'(x_1, w_1])$ and $c_2 \in V(W[z_2, y_2])$, such that $\{c_1, c_2\}$ is a cut in H_0 separating x_1 from w_j. But then, (i) of Lemma 2.4.8 holds, a contradiction. $\Box$

(6) $H_0 - (A'[x_1, w_1] \cup X_1[x_1, y_2) \cup W[z_2, w_m))$ contains a path Y_2^* from y_2 to w_2.

For otherwise, by planarity of H_0, there exist $c_1 \in V(W[z_2, w_m))$ and $c_2 \in V(A'[x_1, w_1]) \cup V(X_1[x_1, y_2))$, such that $\{c_1, c_2\}$ is a 2-cut in H_0 separating y_2 from w_2. Now $c_2 \in X_1[x_1, y_2)$; as otherwise $c_2 \notin A'[x_1, w_1]$ and (i) of Lemma 2.4.8 holds, a contradiction.

Let $w_i \in W(c_1, y_2)$ such that i is minimum, and let $u_i \in N_G(w_i) \cap V(A')$ with $A'[u_2, u_i]$ minimum. Then G has an A'-B' path S from $s \in V(A'(u_i, x_2))$ to $s' \in V(B'[b_1, y_1])$ and internally disjoint from H'; otherwise, $\{u_i, c_1, c_2, y_2, x_2\}$ is a cut in G^* separating w_m from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

By Lemma 2.4.3, S is an edge with $s' \in \{r', y_1\}$. If $s' = r'$ then S, Q contradict (4). So $s' = y_1$. Then $A'[a_1, w_1] \cup W[w_1, z_1] \cup z_1 y_1 \cup A'_0 \cup s's \cup A'[s, a_2]$ and $B'[b_1, q'] \cup Q \cup A'[q, u_i] \cup u_i w_i \cup W[w_i, y_2]$ show that γ is feasible, a contradiction. $\Box$

(7) z_1, x_2 are incident with some finite face of $H' - y_1$.

For otherwise, there exist $k \in \{j+1, \cdots, m\}$ and a vertex $u_k \in V(A'[u_2, x_2))$, such that $w_k u_k \in E(G)$. We choose k with k minimum and choose u_k so that $A'[u_k, a_2]$ is maximal. Clearly, $k = j+1$ or $k = j+2$.

Suppose G has an A'-B' path S from a_2 to $s' \in V(B'[b_1, y_1])$. By (3), $s' \notin B'(b_1, r']$. Moreover, $s' \notin B'(r', y_1]$; otherwise, $S, R, u_k w_k \cup W[w_k, y_2]$, and $X' \cup W[w_j, z_1] \cup z_1 y_1$

force a double cross. So $s' = b_1$. Note that $|V(S)| \geq 3$ as $a_2b_1 \notin E(G)$; so S is contained in an A'-B' bridge N and let n_1, n_2 be the extreme hands of N. Since we forced $s' = b_1$, we see that b_1 is the only foot of N. By Lemma 2.4.3, $V(N \cap A'(u_2, x_2)) = \emptyset$. By (v) of Lemma 2.3.9, $n_1 \notin A'[a_1, u_2)$, and so $n_1 = u_2$. But then, $\{n_1, n_2, b_1\}$ is a cut in G separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Then G has no A'-B' path from a_2 to $B'[b_1, y_1]$ and internally disjoint from H'. Since the degree of a_2 in G is at least 4, G has an edge from a_2 to some $w \in V(W[w_k, w_m))$. We derive $\alpha(A', B') = 2$ by Lemma 2.2.1 and the following paths: the path $A'[a_1, r] \cup R \cup B'[b_1, r']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $B'_1 \cup Q \cup A'[q, u_2] \cup u_2w_2 \cup Y_2^*$ from b_1 to b_2, and the path $a_2w \cup W[w, z_2] \cup z_2y_1 \cup A'_0$ from a_2 to a_0. This contradicts Lemma 2.4.10. □

(8) Let $v_j \in N_G(w_j) \cap V(A')$ with $A'[v_j, a_2]$ is minimal. Then G has two disjoint A'-B' paths from $A'(x_1, v_j)$ to $B'[b_1, y_1]$ and internally disjoint from H'.

For otherwise, there exists $v \in V(G)$ such that $G - v$ does not contain any A'-B' path from $A'(x_1, v_j)$ to $B'[b_1, y_1]$ and internally disjoint from H'. But then, combined with (6), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{v, x_1, y_2, x_2, u, v_j\}$ with $u = y_1$ (when $z_1 \neq z_2$) or $u = z_1$ (when $z_1 = z_2$), $\{a_0, a_1, a_2, b_1, b_2\} \cup V(A'[v_j, x_2]) \subseteq V(G_1)$, and $A'[x_1, v_j] \cup X_1 \subseteq G_2$.

By Lemma 2.1.3, $(G_2, v, x_1, y_2, x_2, u, v_j)$ is not planar. So, clearly, $v \notin A'$, and there exists an A'-B' bridge N with feet n'_1, n'_2 and extreme hands n_1, n_2, such that $v \in N$. By (v) of Lemma 2.3.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to y_2 and internally disjoint from A'. Suppose $v \notin B'$. Then N has a separation (N', N'') of order 1, such that $V(N' \cap N'') = \{v\}$, $n_1, n_2 \in V(N' - N'')$, and $n'_1, n'_2 \in V(N'' - N')$. Now $V(N') = \{n_1, n_2, v\}$; or else, $\{n_1, n_2, v\}$ is a cut in G separating $V(N') - \{n_1, n_2, v\}$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. This implies that $(G_2, v, x_1, y_2, x_2, u, v_j)$ is planar, a contradiction. So $v \in B'$. But then, by (v) of Lemma 2.3.9 and the definition of v, $n'_1 = n'_2 = v$ and there exist $n_1^* \in A'[a_1, n_1]$ and $n_2^* \in A'[n_2, a_2]$, such that $\{n_1^*, n_2^*, v\}$ is a cut in G^* separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\Box$

By (8), let T_1, T_2 be disjoint A'-B' paths from $t_1, t_2 \in A'(x_1, v_j)$ to $t_1', t_2' \in B'[b_1, y_1]$, respectively, which are internally disjoint from H', such that $B'[t_1', t_2']$ is maximal and, subject to this, $A'[t_1, t_2]$ is maximal. We may choose notation so that a_1, t_1, t_2, a_2 occur on A' in order. Then by (4), b_1, t_1', t_2', b_2 occur on B' in order.

(9) $t_1' \in B'[b_1, r']$, and there exist $c_1 \in V(B'[b_1, t_1'])$ and $c_2 \in V(B'[t_2', y_1])$ such that c_1, c_2 are incident with some finite face of G_0'.

First, suppose such $\{c_1, c_2\}$ does not exist. Then G_0' contains a path from a_0 to $B'(t_1', t_2')$ and internally disjoint from B'. This contradicts Lemma 2.2.2 along with the path $A'[a_1, x_1] \cup X' \cup w_j v_j \cup A'[v_j, a_2]$ from a_1 to a_2 and the path $B'[b_1, t_1'] \cup T_1 \cup A'[t_1, t_2] \cup T_2 \cup B'[t_2', y_1] \cup y_1 z_2 \cup W[z_2, y_2]$ from b_1 to b_2.

Now suppose $t_1' \notin B'[b_1, r']$. Then $t_1' \in B'(r', y_1]$. First, assume R is internally disjoint from T_1, T_2. If $r \in A'(t_1, x_2]$ then R, T_1 contradict (4). So $r \in A'[a_1, t_1]$ and, then, R, T_2 contradict the choice of T_1, T_2. So there exists $v \in V(R \cap (T_1 \cup T_2))$, and we choose v so that $R[r', v]$ is minimal. If $v \in V(T_1)$, then $R[r', v] \cup T_1[v, t_1], T_2$ contradict the choice of T_1, T_2; if $v \in V(T_2)$, then $T_1, R[r', v] \cup T_2[v, t_2]$ form a cross, contradicting (4). $\Box$

Now, we further choose c_1, c_2 in (9) so that $B'[c_1, c_2]$ is maximal.

(10) $G_0' - A_0' - B'(b_1, c_1) \cup B'[c_2, y_1]$ contains a path B_0' from b_1 to c_1, and $G_0' - A_0' - B'(b_1, c_2) \cup B'(c_2, y_1]$ contains a path B_0'' from b_1 to c_2.

Suppose B_0' does not exist. Then $B'(b_1, c_1) \neq \emptyset$ and, by planarity of G_0', there exist $b_1' \in V(B'(b_1, c_1))$ and $a_0' \in V(B'[c_2, y_1]) \cup V(A_0')$ such that b_1', a_0' are incident with some finite face of G_0'. If $a_0' \in B'[c_2, y_1]$ then b_1', a_0' contradict the choice of c_1, c_2; if $a_0' \in A_0'$ then $\{b_1', a_0', b_2\}$ is a 3-cut in G_0', contradicting Lemma 2.3.7.

Now suppose B_0'' does not exist. Then by planarity of G_0', there exist $b_1' \in V(B'(b_1, c_2))$ and $a_0' \in V(B'(c_2, y_1]) \cup V(A_0')$, such that b_1', a_0' are incident with some finite face of G_0'. Now, if $a_0' \in V(B'(c_2, y_1])$ then b_1', a_0' or c_1, a_0' contradict the choice of c_1, c_2. So $a_0' \in V(A_0')$. Then $b_1' \in B'(c_1, c_2)$ and $b_1 = c_1$; otherwise, $\{b_1', a_0', b_2\}$ or $\{c_1, a_0', b_2\}$ is a 3-cut in G_0', contradicting Lemma 2.3.7. But now, a_0, b_1, b_1', c_2 are incident with some finite face of G_0'; so $\alpha(A', B') = 0$, a contradiction to Lemma 2.4.10. □

(11) G has no A'-B' path from $B'(b_1, c_1)$ to A', but G has an A'-B' path T from $t' \in B'(c_2, y_1)$ to $t \in A'[x_1, x_2]$.

Note that $c_1 \in B'[b_1, r_1]$, since $c_1 \in B'[b_1, t_1']$ and $t_1' \in B'[b_1, r_1]$. Thus, if G has an A'-B' path from $B'(b_1, c_1)$ to A', it should be an edge ab with $b \in V(B'(b_1, c_1))$ and $a \in V(A'[a_1, x_1]) \cup \{a_2\}$. By (3), $a \in A'[a_1, x_1]$. Now by Lemma 2.2.1, the following paths show $\alpha(A', B') = 2$: the path $A'[a_1, a] \cup ab \cup B'[b_1, b]$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $A_0' \cup B'[q', y_1] \cup Q \cup A'[q, a_2]$ from a_0 to a_2, and the path $B_0' \cup B'[c_1, r'] \cup R \cup A'[r, w_1] \cup W[w_1, y_2]$ from b_1 to b_2. This contradicts Lemma 2.4.10.

Now the path T must exist; otherwise $\{b_1, c_1, c_2, y_1, b_2\}$ is a cut in G^* separating a_0 from $\{a_1, a_2\}$, a contradiction. □

We choose T in (11) so that $A'[t, a_2]$ is minimal. Then

(12) $t \neq a_2$, T is internally disjoint from T_1, T_2, and $t = u_2 = v_j$.

First, suppose there exists $v \in V(T \cap (T_1 \cup T_2))$, and choose v with $T[v, t']$ minimal. If $v \in T_1$ then $T_1[t_1, v] \cup T[v, t'], T_2$ contradict (4); if $v \in T_2$ then $T_1, T_2[t_2, v] \cup T[v, t']$ contradict the choice of T_1, T_2. So T is internally disjoint from T_1, T_2.

Now suppose $t = a_2$. By Lemma 2.2.1, the following paths show that $\alpha(A', B') = 2$: the path $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $T \cup B'[t', y_1] \cup A_0'$ from a_2 to a_0, and the path $B_0'' \cup B'[t_2, c_2] \cup T_2 \cup A'[t_2, u_2] \cup u_2w_2 \cup W[w_2, y_2]$ from b_1 to b_2. This contradicts Lemma 2.4.10.

By (4), $t \in A'[t_2, a_2)$. By the choice of T_1, T_2, $t \notin A'[t_2, v_j)$. By Lemma 2.4.3, we have $t \notin A'(u_2, a_2)$, and so $t = u_2 = v_j$. $\square$

(13) $t_1 \in A'[a_1, w_1)$.

For otherwise, $t_1 \in A'[w_1, v_j)$. Suppose that G has no A'-B' path from $A'(x_1, w_1)$ to $B'[b_1, y_1]$ and internally disjoint from H'. By (7) and $u_2 = v_j$ in (12), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{x_1, w_1, u_2, u, x_2, y_2\}$ with $u = y_1$ (when $z_1 \neq z_2$) or $u = z_1$ (when $z_1 = z_2$), $\{a_0, a_1, a_2, b_1, b_2\} \cup V(A'[u_2, x_2]) \subseteq V(G_1)$, $X_1 \cup X_2 \subseteq G_2$, and $(G_2, x_1, w_1, u_2, u, x_2, y_2)$ is planar. This contradicts Lemma 2.1.3.

So G has an A'-B' path T_0 from $t_0 \in A'(x_1, w_1)$ to $t_0' \in B'[b_1, y_1]$ and internally disjoint from H'. If T_0 is disjoint from T_1, T_2 then either T_0, T_2 contradict the choice of T_1, T_2, or T_0, T_1 contradict (4). So there exists $v \in V(T_0 \cap (T_1 \cup T_2))$, and we choose v with $T_0[v, t_0']$ minimal. If $v \in T_1$ then $T_1[t_1, v] \cup T_0[v, t_0'], T_2$ contradict the choice of T_1, T_2; if $v \in T_2$ then $T_1, T_2[t_2, v] \cup T_0[v, t_0']$ contradict (4). $\square$

Now, by (13) and Lemma 2.2.1, the following paths show $\alpha(A', B') = 2$: the path from $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2, the path $A'[t, a_2] \cup T \cup B'[t', y_1] \cup A_0'$ from a_2 to a_0, and the path $B_0'' \cup B'[t_2, c_2] \cup T_2 \cup A'[w_1, t_2] \cup W[w_1, y_2]$ from b_1 to b_2. This contradicts Lemma 2.4.10. $\square$

Lemma 2.4.12 *There is no fat A'-B' connector in γ.*

Proof. For, otherwise, (i) or (ii) of Lemma 2.4.8 holds. Then

(a) if (i) of Lemma 2.4.8 holds then, by Lemma 2.4.9, we may choose the 2-cut $\{z_1', z_2'\}$ so that $z_1' \in A'[x_1, w_1]$.

(b) if (i) of Lemma 2.4.8 does not hold but (ii) of Lemma 2.4.8 holds then, by Lemma 2.4.11, $N_G(y_1) \cap V(H_0) \subseteq V(W[w_1, w_2])$ and let $z_1' := w_1$ and $z_2' := z_1$.

(1) $z_2' \notin V(X_2)$.

For, suppose $z_2' \in V(X_2)$. Since $z_1 \notin V(X_2)$ by Lemma 2.4.7, (a) holds. Then $z_1' = x_1$; or else, it contradicts Lemma 2.4.1 that $H' - A'(x_1, x_2)$ contains disjoint paths from y_1, y_2 to x_1, x_2, respectively. But now, $\{x_1, y_2, z_2'\}$ is a cut in G^* separating $X_1(x_1, y_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

By (1), $w_m \in W(z_2', y_2)$. Let $h \in \{2, \cdots, m\}$ be minimum with $w_h \in W(z_2', y_2)$, and let $u_h \in N_G(w_h) \cap V(A'[q, x_2])$ with $A'[q, u_h]$ minimal. Let $Y_2 := W[y_2, w_h] \cup w_h u_h$, which is a path from y_2 to u_h.

(2) $G[H_0 + y_1] - A'(x_1, w_1]$ contains a path Y_1 from y_1 to x_1 and disjoint from Y_2.

Let $v \in N_G(y_1) \cap V(H_0)$ such that $v \notin A'$. Them $v \notin W[w_h, y_2]$. If $H_0 - W[w_h, y_2] - A'(x_1, w_1]$ contains a path Y from v to x_1 then $Y \cup v y_1$ gives the desired Y_1. So assume such Y does not exist. Then, by the planar structure of H_0, there exist $z_1'' \in V(A'[x_1, z_1']), z_2'' \in V(W[w_h, y_2])$ such that z_1'', z_2'' are incident with some finite face of H_0, and $\{z_1'', z_2''\}$ is a 2-cut in H_0. But then, $\{z_1'', z_2''\}$ contradicts the choice of $\{z_1', z_2'\}$. $\square$

Let $Y_1' := Z_2 \cup W[z_2, w_m] \cup w_m v_m$, which is a path from y_1 to x_2. Then

(3) $H_0 - Y_1'$ has a path Y_2' from y_2 to z_1' and internally disjoint from A'.

For otherwise, by the planar structure of H_0, we may assume there exist $z_1'' \in V(A'[x_1, z_1'))$, $z_2'' \in V(W[z_2, w_m])$ such that z_1'', z_2'' are incident with some finite face of H_0, and $\{z_1'', z_2''\}$ is a 2-cut in H_0. But $\{z_1'', z_2''\}$ contradicts the choice of $\{z_1', z_2'\}$. $\square$

Now, the following statement holds to avoid forming a double cross with Y_1', Y_2':

(4) G has no disjoint A'-B' paths from $c, d \in V(A')$ to $c', d' \in V(B'[b_1, y_1])$, respectively, and internally disjoint from $A' \cup B' \cup H'$, such that $c \in V(A'[a_1, z_1'))$, $d \in V(A'(c, x_2))$, and b_1, d', c', y_1 occur on B' in order.

(5) If $u_h \neq x_2$ and G has an A'-B' path S from $s \in A'(u_h, x_2]$ to $s' \in B'[b_1, y_1]$ and internally disjoint from H', then $b_1 = r_1 = r' = s'$ and S is an edge from s to s'.

Firs, $S \cap R = \emptyset$; otherwise, S, R are contained in some A'-B' bridge, which contradicts (v) of Lemma 2.3.9 due to the path $u_h w_h \cup W[w_h, y_2]$ from u_h to y_2. Now, $s' \in B'[b_1, r']$; otherwise, S, R, Y_1, Y_2 form a doublecross as we assume $u_h \neq x_2$. Thus, G has no A'-B' path from $A'(u_h, x_2]$ to $B'(r', y_1]$, which further implies that $S \cap Q = \emptyset$.

We claim $b_1 = r_1$ and so $a_1 = x_1$ by (iii) of Lemma 2.4.4. For, suppose $b_1 \neq r_1$. Then $s' \neq b_1$; otherwise, $s = x_2 = a_2$, and $S = a_2 b_1$, a contradiction. But then, $A'[a_1, r] \cup R \cup B'[s', r'] \cup S \cup A'[s, a_2] \cup Y_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[q, u_h] \cup Y_2 \cup B'[y_2, b_2]$ show that γ is feasible, a contradiction. (Recall B'_1, A'_0 from Lemma 2.4.10.)

Now suppose $r_1 \neq r'$. By Lemma 2.4.2, there exist an A-B core H'' with feet r_1, r_2 and $r' \in B'(r_1, r_2)$, and an A'-B' bridge M with extreme hands l_0, r_0 and feet l'_0, r'_0, such that R is internally disjoint from M, $l_0 = r_0 = x_i$ for some $i \in [2]$, and $r' \in B'(l'_0, r'_0)$. Since G has no A'-B' path from $A'(u_h, x_2]$ to $B'(r', y_1]$, then $i = 1$, x_1 is an extreme hand of H'', and S is internally disjoint from M. If $s' = r'$ then let P^* be the path from l'_0 to r'_0 in M and internally disjoint from A', B'; now $A'[a_1, r] \cup R \cup S \cup A'(u_h, a_2] \cup Y_1 \cup A'_0$ and $B'[b_1, l'_0] \cup P^* \cup B'[r'_0, q'] \cup Q \cup A'[q, u_h] \cup Y_2$ show that γ is feasible, a contradiction. Thus, $s' \in B'[r_1, r')$ and $s = x_2$ (by the definition of r'). Now, we see that S is not contained in an A'-B' bridge. For otherwise, by (ii) of Lemma 2.3.9, S is contained in H'', which further implies x_2 is an extreme hand of H''. So H'' is a main core of A, B, a contradiction to Lemma 2.3.8. So $S = x_2 s'$. If $s' \in B'(r_1, r')$ then $S \in E(H'')$, which implies that x_2 is an extreme hand of H'', still a contradiction to Lemma 2.3.8. So $s' = r_1$ and $S = x_2 b_1$, which implies $a_2 \neq x_2$, a contradiction to Lemma 2.4.8.

Therefore, $b_1 = r_1 = r' = s'$. To complete the proof of (5), we need to prove that $S = ss'$. For, suppose $S \neq ss'$. Then S is contained in some A'-B' bridge N, and let n_1, n_2 be the extreme hands of N. Note that $V(N \cap B') \subseteq \{b_1\}$, as $b_1 = r_1 = s' = r'$ for any choice of S. Moreover, by Lemma 2.4.3, $V(N \cap A'(u_h, x_2)) = \emptyset$. Hence, $n_1 \in A'[x_1, u_h]$ and $n_2 = x_2$. By (v) of Lemma 2.3.9, $H' - y_1$ does not have a path from $A'(n_1, n_2)$ to y_2 and internally disjoint from A'. So, by the existence of path Y_2, $n_1 \notin A'[x_1, u_2)$. So $n_1 = u_h$.

But then, $\{n_1, n_2, b_1\}$ is a cut in G^* separating N from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

(6) $x_1 \neq z_1'$, and $b_2 = y_2$.

First, suppose $x_1 = z_1'$. Since $w_1 \neq x_1$ then (a) holds. Now G has an A'-B' path from $A'(u_h, x_2)$ to $B'[b_1, y_1]$ internally disjoint from $H' - y_1$; otherwise, $\{x_1, z_2', u_h, x_2, y_2\}$ is a cut in G^* separating $\{a_0, a_1, a_2, b_1, b_2\}$ from $V(X_1 \cup X_2)$, a contradiction. Hence, $A'(u_h, x_2) \neq \emptyset$ and, by (5), $b_1 = r_1 = r'$ and $a_1 = x_1$ (by (iii) of Lemma 2.4.4). But then, G has a separation (G_1, G_2) of order 6, such that $V(G_1 \cap G_2) = \{x_1, z_2', u_h, x_2, y_2, b_1\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $V(X_1 \cup X_2) \subseteq V(G_2)$, and $(G_2, x_1, y_2, x_2, b_1, u_h, z_2')$ is planar, a contradiction to Lemma 2.1.3.

Now suppose $b_2 \neq y_2$. By Lemma 2.4.8, $N_G(b_2) = \{y_2, x_1\}$ and $a_1 \neq x_1$. Let $a_1 b' \in E(G)$ with $b' \in V(B'(b_1, r_1]) \cup V(B'[y_2, b_2))$. By (i) of Lemma 2.4.4, $b' \in B'(b_1, r_1]$. Since $x_1 \neq z_1'$, we have $\alpha(A', B') = 2$ by Lemma 2.2.1 and the followin paths: the path $A_0' \cup Y_1' \cup A'[x_2, a_2]$ from a_0 to a_2, the path $B_1' \cup Q \cup A'[z_1', q] \cup Y_2' \cup B'[y_2, b_2]$ from b_1 to b_2, the path $a_1 b' \cup B'[b_1, b']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup e$ from a_1 to b_2. This contradicts Lemma 2.4.10. $\square$

(7) G has an A'-B' path from $A'[a_1, z_1')$ to $B'(b_1, y_1]$ and internally disjoint from H'.

For, suppose (7) fails. Then by Lemma 2.4.10 and by (5) and (6) ($b_2 = y_2$), if (a) holds then $\{b_1, b_2, z_1', z_2', u_h\}$ is a cut in G^* separating a_1, a_2 from a_0, a contradiction; if (b) holds then $\{b_1, b_2, z_1', y_1, u_h\}$ (when $z_1 \neq w_2$) or $\{b_1, b_2, z_1', z_1, u_h\}$ (when $z_1 = w_2$) is a cut in G separating a_1, a_2 from a_0, a contradiction. $\square$

(8) If $u_h \neq x_2$, then G has no A'-B' path from $A'(u_h, x_2]$ to $B'[b_1, y_1]$ and internally disjoint from H'.

For, otherwise, it follows from (5) that $b_1 = r_1 = r' = s'$ and G has an edge sb_1 with $s \in V(A(u_h, x_2])$. So $s \neq a_2$. Now sb_1 and a path from (7) contradict (4). $\square$

(9) G has disjoint A'-B' paths from $A'[a_1, z_1')$ to $B'[b_1, y_1]$ and internally disjoint from H'.

For otherwise, there exists a vertex $v \in V(G)$ such that $G - v$ has no A'-B' path from $A'[a_1, z_1')$ to $B'[b_1, y_1]$ and internally disjoint from H'. Then by (8), there exists a separation (G_1, G_2) in G such that $V(G_1 \cap G_2) = \{v, z_1', u, u_h\}$ (with $u = z_2'$ if (a) holds and $u = y_1$ if (b) holds), $b_1, a_0 \in V(G_1)$, and $a_1, a_2, b_2 \in V(G_2)$.

Suppose $(G_2, v, z_1', u, u_h, a_2, b_2, a_1)$ is planar. If $v = a_1, u_h = a_2$ then $\{v, z_1', u, u_h, b_2\}$ is a cut in G^* separating $V(X_1 \cup X_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v \neq a_1, u_h = a_2$ or $v = a_1, u_h \neq a_2$, then Lemma 2.1.3 applies; if $v \neq a_1, u_h \neq a_2$, then Lemma 2.1.4 applies.

So $(G_2, v, z_1', u, u_h, a_2, b_2, a_1)$ is not planar. Clearly, $v \notin A'$, and there exists an A'-B' bridge N with feet n_1', n_2' and extreme hands n_1, n_2, such that $v \in N$. By (v) of Lemma 2.3.9, $H' - y_1$ does not contain a path from $A'(n_1, n_2)$ to y_2 and internally disjoint from A'. Suppose $v \notin B'$. Then N has a separation (N', N'') of order 1, such that $V(N' \cap N'') = \{v\}$, $n_1, n_2 \in V(N' - N'')$, and $n_1', n_2' \in V(N'' - N')$. Now $V(N') = \{n_1, n_2, v\}$; or else, $\{n_1, n_2, v\}$ is a cut in G separating $V(N') - \{n_1, n_2, v\}$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. This implies that $(G_2, v, z_1', u, u_h, a_2, b_2, a_1)$ is planar, a contradiction. So $v \in B'$. But then, by (v) of Lemma 2.3.9 and the definition of v, $n_1' = n_2' = v$ and there exist $n_1^* \in A'[a_1, n_1]$ and $n_2^* \in A'[n_2, a_2]$, such that $\{n_1^*, n_2^*, v\}$ is a cut in G^* separating $V(N)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

By (9), let T_1, T_2 be disjoint A'-B' paths from $t_1, t_2 \in A'[a_1, z_1')$ to $t_1', t_2' \in B'[b_1, y_1]$, such that a_1, t_1, t_2, a_2 occur on A' in order, T_1, T_2 are internally disjoint from H' and, subject to this, $A'[t_1, t_2] \cup B'[t_1', t_2']$ are maximal. Then by (4), b_1, t_1', t_2', y_1 occur on B' in order.

(10) $t_1' \in B'[b_1, r']$, $t_2' \notin B'(q', y_1]$, and Q is internally disjoint from T_1, T_2.

Suppose Q is not internally disjoint from T_j for some $j \in [2]$, then Q, T_j are contained in some A'-B' bridge. But then, the existence of the path from z'_1 to y_2 in $H' - y_1$ contradicts (v) of Lemma 2.3.9.

So Q is internally disjoint from T_1, T_2. Hence, by (4), $t'_2 \notin B'(q', y_1]$. Now suppose $t'_1 \in B'(r', t'_2)$. If $R \cap (T_1 \cup T_2) = \emptyset$, then R, T_2 contradict the choice of T_1, T_2 (when $r \in A'[a_1, t_1]$) or T_1, R contradict (4) (when $r \in A'(t_1, q)$). So there exists $u \in V(R \cap (T_1 \cup T_2))$, and we choose u so that $R[r', u]$ is minimal. If $u \in T_1$, then $R[r', u] \cup T_1[u, t_1], T_2$ contradict the choice of T_1, T_2. If $u \in T_2$, then $T_1, R[r', u] \cup T_2[u, t_2]$ contradict (4). $\square$

We let Q_0 be an A'-B' path from $q_0 \in A'(z'_1, a_2]$ to $q'_0 \in B'[b_1, y_1]$ and internally disjoint from H', such that $B'[q'_0, y_1]$ is minimal. By the existence of Q, $q'_0 \in B'[q', y_1]$.

(11) No finite face of G'_0 is incident with both a vertex of $B'[b_1, t'_1]$ and a vertex of $B'[q'_0, y_1]$.

For, suppose $c_1 \in V(B'[b_1, t'_1])$ and $c_2 \in V(B'[q'_0, y_1])$ such that c_1, c_2 are incident with a finite face of G'_0. We choose c_1, c_2 so that $B'[c_1, c_2]$ is maximal. Since $t'_1 \in B'[b_1, r']$, $c_1 \in B'[b_1, r']$. We may further assume $c_1 \in B'[b_1, r_1]$; otherwise, $r' \neq r_1$, $c_1 \in B'(r_1, r']$, and by (iii) of Lemma 2.4.2, $r' \in B'(r_1, r_2)$ for some $r_2 \in V(B'(r', y_1])$ and r', r_1, r_2 are incident with some finite face of G'_0, implying $c_1 \in B'[b_1, r_1]$ by the choice of c_1, c_2, a contradiction.

Note that G has an A'-B' path T_3 from $t'_3 \in B'(b_1, c_1) \cup B'(c_2, y_1)$ to $t_3 \in A'$, to avoid the cut $\{b_1, b_2, c_1, c_2, y_1\}$ in G^*, separating a_0 from $\{a_1, a_2\}$.

Note that $t'_3 \in B'(c_2, y_1)$. For, suppose $t'_3 \in B'(b_1, c_1)$. Then $t'_3 \in B'(b_1, r_1)$ and, by the choice of T_1, T_2 and by (4) and (8), we have $t_3 = u_h = a_2$. Thus, $A'[a_1, t_1] \cup T_1 \cup B'[t'_3, t'_1] \cup T_3 \cup Y'_1 \cup A'_0$ and $B'_1 \cup Q \cup A'[z'_1, q] \cup Y'_2$ show that γ is feasible, a contradiction.

Moreover, $t_3 = z'_1$, as $t_3 \notin A'(z'_1, a_2]$ (by the choice of Q_0), and $t_3 \notin A'[a_1, z'_1)$ (so that T_3, Q_0 do not contradict (4)).

If $G'_0 - B'[t'_1, q'_0] - A'_0$ contains a path B^*_3 from b_1 to t'_3, then $A'[a_1, t_1] \cup T_1 \cup B'[t'_1, q'_0] \cup Q_0 \cup A'[q_0, a_2] \cup Y'_1 \cup A'_0$ and $B^*_3 \cup T_3 \cup Y'_2$ show that γ is feasible, a contradiction.

So such B_3^* does not exist. Then, by the maximality of $B'[c_1, c_2]$, there exists $c_3 \in V(A'_0)$ such that $\{c_2, c_3\}$ is a cut in G'_0 separating b_1 from t'_3, and there does not exist any A'-B' bridge with one foot in $B'[b_1, c_2)$ and another in $B'(c_2, y_1]$. Hence, $\{z'_1, c_2, c_3, y_1\}$ is a cut in G^* separating t'_3 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(12) $G'_0 - B'(b_1, t'_1] - (B'[q'_0, y_1] \cup A'_0)$ contains a path B_1^* from b_1 to $B'(t'_1, q'_0)$.

For otherwise, $b_1 \neq t'_1$, and there exist $c_1 \in V(B'(b_1, t'_1])$ and $c_2 \in V(B'[q'_0, y_1]) \cup V(A'_0)$ such that c_1, c_2 are incident with a finite face of G'_0. By (11), $c_2 \in A'_0$. By Lemma 2.3.7, $c_1 \notin B'(b_1, r_1]$. So $c_1 \in B'(r_1, r']$ as $t'_1 \in B'[b_1, r']$. Hence, by (iv) of Lemma 2.4.2, $c_2 = a_0$, $b_1 = r_1$, and $\alpha(A', B') = 0$, contradicting Lemma 2.4.10. □

(13) If (a) holds then $H' - y_1 - z'_2 - X_1[x_1, y_2)$ has a path Y_2^* from z'_1 to y_2 and internally disjoint from A'.

For otherwise, there exists $u \in V(A'[x_1, z'_1) \cup X_1[x_1, y_2))$, such that u, z'_2 are incident with a finite face of $H' - y_1$. By the choice of $\{z'_1, z'_2\}$, $u \in V(X_1(x_1, y_2))$. Now $\{u, z'_2, u_h, x_2, y_2\}$ is a cut in G^* separating X_2 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. □

(14) If (a) holds then $H' - y_1 - A'(x_1, z'_1) - W[z'_2, y_2]$ has a path X^* from x_1 to z'_1; if (b) holds then $H' - y_1 - A'(x_1, z'_1) - W[z_2, y_2]$ has a path X^* from x_1 to z'_1.

For otherwise, let $v = z'_2$ when (a) holds; and let $v = z_2$ when (b) holds. Then there exist $z''_1 \in V(A'(x_1, z'_1))$ and $z''_2 \in V(W[v, y_2])$ such that z''_1, z''_2 are incident with a finite face of H_0. Hence, (a) holds, and $\{z''_1, z''_2\}$ contradicts the choice of $\{z'_1, z'_2\}$. □

(15) $G - T_1 - Q_0$ has no A'-B' path from $A'(t_1, z'_1]$ to $B'(t'_1, q'_0)$.

For, suppose $G - T_1 - Q_0$ has an A'-B' path T from $t \in V(A'(t_1, z'_1])$ to $t' \in V(B'(t'_1, q'_0))$. When (a) holds, we let B^* be the path from b_1 to b_2 in $B_1^* \cup B'(t'_1, q'_0) \cup T \cup A'[t, z'_1] \cup Y_2^*$; when (b) holds, we let B^* be the path from b_1 to b_2 in $B_1^* \cup B'(t'_1, q'_0) \cup T \cup W[t, y_2]$. By

Lemma 2.2.1, the following paths show that $\alpha(A', B') = 2$: the path B^* from b_1 to b_2, the path $A'[q_0, a_2] \cup Q_0 \cup B'[q_0', y_1] \cup A_0'$ from a_2 to a_0, the path $A'[a_1, t_1] \cup T_1 \cup B'[b_1, t_1']$ from a_1 to b_1, and the path $A'[a_1, x_1] \cup X_1$ from a_1 to b_2. This contradicts Lemma 2.4.10. $\Box$

(16) $t_2' = q_0'$, $t_1' = r'$, and G has an A'-B' path R^* from r' to $A'(x_1, z_1')$.

For, suppose $t_2' \neq q_0'$. By (15), T_2, Q_0 are contained in an A'-B' bridge. But the existence of the path from z_1' to y_2 in $H' - y_1$ contradicts (v) of Lemma 2.3.9.

Note that G has an A'-B' path from r' to $A'(x_1, z_1')$; for otherwise, $R \cap T_2 = \emptyset$, and R, T_2 contradicts (4).

Next $t_1' = r'$. For otherwise, $r' \in B'(t_1', q_0')$. Now, by (15), $R^* \cap (T_1 \cup Q_0) \neq \emptyset$. By the definition of r', $R^* \cap T_1 = \emptyset$. Thus, R^*, Q_0 are contained in some A'-B' bridge. But then, the path from z_1' to y_2 in $H' - y_1$ contradicts (v) of Lemma 2.3.9. $\Box$

Now, the path $A'[a_1, x_1] \cup X^* \cup A'[z_1', a_2]$ from a_1 to a_2 and the path $B'[b_1, r'] \cup R \cup A'[r, t_2] \cup T_2 \cup B'[t_2', y_1] \cup Z_2 \cup W[z_2, y_2]$ from b_1 to b_2 show that G_0' does not contain a path from $B'(t_1', t_2')$ to a_0 and internally disjoint from B'; or else, it contradicts (i) of Lemma 2.2.2. So, there exist $c_1 \in B'[b_1, t_1']$ and $c_2 \in B'[t_2', y_2]$, such that c_1, c_2 are incident with some finite face of G_0', a contradiction to (11). $\Box$

2.5 Slim connectors

In this section, we let $\gamma := (G, a_0, a_1, a_2, b_1, b_2)$, and assume that γ is infeasible and no ideal frame in γ admits a fat connector (seen at Figure 2.11).

Recall that $b_1b_2 \notin E(G)$, $a_ib_j \notin E(G)$ for $i = 0, 1, 2$ and $j = 1, 2$, and $G^* := G + b_1b_2 + \{a_ib_j : i = 0, 1, 2 \text{ and } j = 1, 2\}$ is 6-connected. Let A, B be an ideal a_0-frame in γ. Let $G_0 := G - A$. By Lemma 2.1.6 and the structure of slim connectors, G_0 has a disk representation with B and a_0 occurring on the boundary of the disk, and any A-B path in G is induced by a single edge.

Lemma 2.5.1 *Let $a_{-1} := a_2$ and $a_3 := a_0$. Then the following statements hold:*

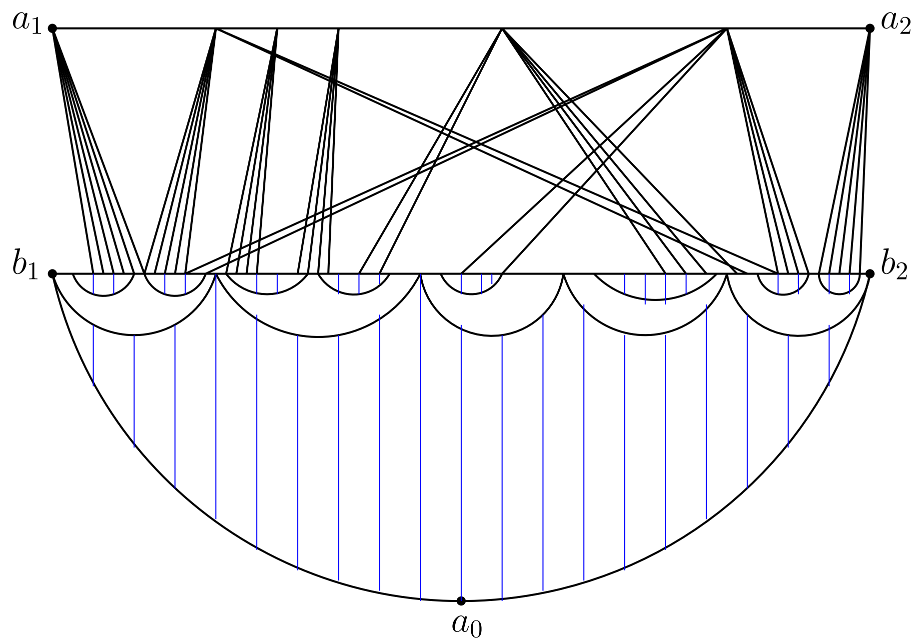

Figure 2.11: An ideal frame with only slim connectors

(i) *G cannot be obtained from a planar graph H by identifying two vertices of H, such that b_1, b_2 and two of $\{a_0, a_1, a_2\}$ are incident with a face of H.*

(ii) *For any $i \in \{0, 1, 2\}$, $(G - a_{i-1}, a_i, b_1, a_{i+1}, b_2)$ or $(G - a_{i+1}, a_i, b_1, a_{i-1}, b_2)$ is not planar.*

(iii) *There do not exist a permutation π of $\{0, 1, 2\}$, a graph H and distinct vertices $s, t, s', t' \in V(H)$, such that $(H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2)$ is planar, and G is obtained from H by identifying s with s' and t with t', respectively.*

Proof. Let $n = |V(G)|$. Since G^* is 6-connected, $|E(G)| \geq 3n - 7$. First, we see that (i) holds. For, otherwise, there exist $i \in \{0, 1, 2\}$, graph H with $(H, a_{i-1}, b_1, a_{i+1}, b_2)$ planar, and distinct $s, s' \in V(H)$, such that G is isomorphic to the graph obtained from H by identifying s with s'. Then $|E(H)| \geq |E(G)| \geq 3n-7$, and $H' := H + \{a_{i-1}b_1, a_{i-1}b_2, a_{i+1}b_1, a_{i+1}b_2, b_1b_2\}$ is planar. However, $|E(H')| \geq 3n - 2 = 3|V(H')| - 5$, a contradiction.

Now suppose (ii) fails. Then for some $i \in \{0, 1, 2\}$, both $(G - a_{i-1}, a_i, b_1, a_{i+1}, b_2)$ and $(G - a_{i+1}, a_i, b_1, a_{i-1}, b_2)$ are planar. Without loss of generality, we assume $i = 0$ and that

$d_G(a_1) \leq d_G(a_2)$. Let $G' := G + \{a_2b_1, a_2b_2, a_0b_1, a_0b_2, b_1b_2\}$. Then $G' - a_1$ is planar. Since G^* is 6-connected, $d_{G'}(a_2) \geq d_G(a_1) + 2$, $d_{G'}(a_0) \geq 6$, $d_{G'}(b_j) \geq 5$ for $j \in [2]$, and $d_{G'}(x) \geq 6$ for all $x \in V(G') \setminus \{a_0, a_1, a_2, b_1, b_2\}$. Hence,

$$|E(G' - a_1)| = (6(n-5) + 6 + 5 + 5 + 2)/2 = 3n - 6 = 3|V(G' - a_1)| - 3,$$

contradicting the planarity of $G' - a_1$.

Finally, suppose (iii) fails. So there exists a permutation π of $\{0, 1, 2\}$, a graph H and distinct vertices $s, t, s', t' \in V(H)$, such that $(H, a_{\pi(0)}, b_1, a_{\pi(1)}, s, t, s', t', a_{\pi(2)}, b_2)$ is planar, and G is obtained from H by identifying s with s' and t with t', respectively. Now $|E(H)| \geq |E(G)| \geq 3n - 7$, $a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s' \notin E(H)$, and $H' := H + \{b_1a_{\pi(0)}, b_1a_{\pi(1)}, b_2a_{\pi(0)}, b_2a_{\pi(2)}, a_{\pi(0)}a_{\pi(1)}, a_{\pi(0)}a_{\pi(2)}, a_{\pi(0)}t, a_{\pi(0)}s'\}$ is planar. Thus, $|V(H')| = n + 2$ and $|E(H')| \geq 3n + 1 = 3(n + 2) - 5$, contradicting planarity of H'. $\square$

We now investigate the edges between A and B. Let $a'b', a''b'' \in E(G)$ with $a', a'' \in V(A)$ and $b', b'' \in V(B)$ all distinct. We say that $a'b', a''b''$ form a *cross* (w.r.t. A, B) if a_1, a', a'', a_2 occur on A in order, and b_1, b'', b', b_2 occur on B in order. We say that $a'b', a''b''$ are *parallel* if a_1, a', a'', a_2 occur on A in order, and b_1, b', b'', b_2 occur on B in order.

Two sets of edges of G between A and B play critical roles in the remainder of this section. For $i = 5, 6, 7$, let $e_i = a_ib_i \in E(G)$ with $a_i \in V(A)$ to $b_i \in V(B)$; we say that (e_5, e_6, e_7) is a *3-edge configuration* if $b_6 \in B(b_5, b_7)$ and $a_1, a_2, a_6 \notin A[a_5, a_7]$. For $i = 3, 4, 5, 6, 7$, let $e_i = a_ib_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$; we say that $(e_3, e_4, e_5, e_6, e_7)$ is a *5-edge configuration* (seen at Figure 2.12) if

- (e_5, e_6, e_7) is a 3-edge configuration,
- $A[a_5, a_7] \subseteq A(a_3, a_4)$, and
- $b_3, b_4 \in B(b_j, b_5) \cap B(b_j, b_7)$ for some $j \in [2]$.

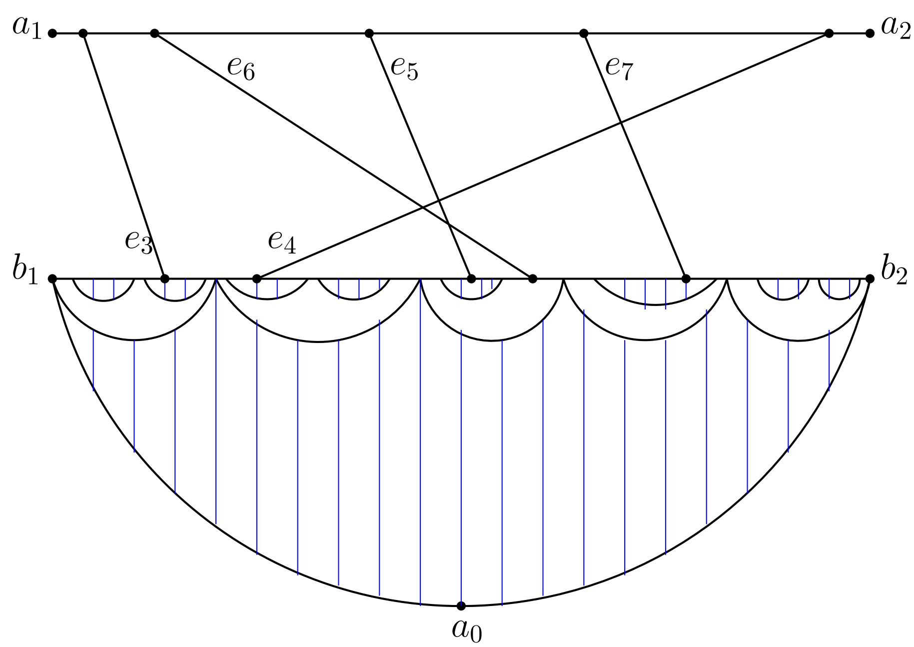

Figure 2.12: $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration

Lemma 2.5.2 *There exists a 5-edge configuration.*

Proof. (1) For $i \in [2]$, G has a cross from $A - a_i$ to B.

For, suppose G has no cross from $A - a_i$ to B and, without loss of generality, let $i = 2$. Let $a'b' \in E(G)$ with $a' \in V(A[a_1, a_2)$ and $b' \in V(B[b_1, b_2])$, such that $B[b', b_2]$ is minimal. Then G has an edge from a_2 to $B[b_1, b')$, as otherwise, $(G, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. Let $a_2u_i \in E(G)$, with $u_i \in V(B[b'_1, b'))$ for $i \in [2]$, such that $B[u_1, u_2]$ is maximal and b_1, u_1, u_2, b_2 occur on B in order.

Then there exists $ab \in E(G)$ with $b \in V(B(u_1, u_2))$ and $a \in V(A[a_1, a_2))$. For, otherwise, let H be obtained from G by splitting a_2 to s, s', such that H has no edge from $B[u_1, u_2]$ to s' and no edge from $B[b', b_2]$ to s. Now (H, a_1, b_2, a_0, b_1) is planar and G can be obtained from H by identifying s and s', contradicting (i) of Lemma 2.5.1.

We see that $a = a_1$. For, otherwise, let $a_1b^* \in E(G)$ with $b^* \neq b$. Since G has no cross from $A - a_2$ to B, $b^* \in B(b_1, b)$. Now, $(a_1b^*, u_1a_2, ab, u_2a_2, a'b')$ is a 5-edge configuration.

So all edges from $B(u_1, u_2)$ to $A[a_1, a_2)$ end with a_1. But now, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1. $\square$

We let $b_1', b_2' \in B[b_1, b_2]$, such that b_1, b_1', b_2', b_2 occur on B in order, G has an edge from b_i' to A for each $i \in [2]$, and subject to this, $B[b_1', b_2']$ is maximal. By relabelling notation, we may assume that

(2) G has no edge from b_1' to $A(a_1, a_2)$, and has an edge $e_3 := b_1'a_1$.

First, suppose there exist $b_i'a_i' \in E(G)$ with $a_i' \in V(A(a_1, a_2))$ for each $i \in [2]$. Since $d_G(a_i) \geq 4$ for $i \in [2]$, there exists $a_ib_i'' \in E(G)$ with $b_i'' \in V(B(b_1', b_2'))$. Now $b_1'a_1', b_2'a_2'$, $b_1''a_1, b_2''a_2$ form a double cross in γ, a contradiction.

Thus, for some $i \in [2]$, G has no edge from b_i' to $A(a_1, a_2)$. By symmetry, we may assume $i = 1$ and $b_1'a_1 \in E(G)$. $\square$

By (1), there exist $e_4 = a_4b_4, e_5 = a_5b_5 \in E(G)$ with $a_4, a_5 \in V(A(a_1, a_2])$ and $b_4, b_5 \in V(B[b_1', b_2])$, such that e_4, e_5 form a cross, and b_1, b_4, b_5, b_2 occur on B in order. We further choose e_4, e_5 so that $B[b_1', b_4] \cup A[a_1, a_5]$ is minimal and, subject to this, $B[b_5, b_2] \cup A[a_4, a_2]$ is minimal. Then

(3) G has no edge from $B[b_1, b_4)$ to $A(a_5, a_2]$, no edge from $A(a_1, a_5)$ to $B(b_4, b_2]$, no edge from b_4 to $A(a_4, a_2]$, and no edge from a_5 to $B(b_5, b_2]$.

To avoid forming a double cross with e_4, e_5,

(4) G has no cross from $B[b_1, b_4]$ to $A[a_1, a_5]$ or from $B[b_5, b_2]$ to $A[a_4, a_2]$.

(5) G has no edge $B(b_5, b_2]$ to $A(a_1, a_4)$, or no edge from $B(b_4, b_5)$ to $A(a_1, a_4) - a_5$.

For, suppose there exists $ab, a'b' \in E(G)$ with $b \in V(B(b_5, b_2])$, $a \in V(A(a_1, a_4))$, $b' \in V(B(b_4, b_5))$ to $a' \in V(A(a_1, a_4) - a_5)$. By (3), $a, a' \in A(a_5, a_4)$. Now $(e_3, e_4, b'a', e_5, ba)$ is a 5-edge configuration. $\square$

Let $e_5' = a_5b_5' \in E(G)$ with $b_5' \in V(B(b_4, b_5])$ such that $B[b_5', b_2]$ is maximal. If G has an edge e from $B(b_5', b_5)$ to $A - a_5$, then (e_3, e_4, e_5', e, e_5) is a 5-edge configuration. Hence, we may assume that

(6) G has no edge from $B(b_5', b_5)$ to $A - a_5$.

We may also assume that

(7) G has no cross from $B[b_5', b_2]$ to $A(a_5, a_2]$ not involving the possible edge $a_4 b_5'$.

For, suppose G has a cross $e' = a'b', e'' = a''b''$ avoiding a_4b_5', with $a', a'' \in V(A(a_5, a_2])$, $b', b'' \in V(B[b_5', b_2])$, and a_5, a', a'', a_2 on A in order. Then $a'' \in A(a_5, a_4]$, to avoid the double cross e_4, e_5', e', e''. Hence, we may assume $b'' = b_5'$; as otherwise, $(e_3, e_4, e_5', e'', e')$ is a 5-edge configuration. Then $a'' \in A(a_5, a_4)$, as $e'' \neq a_4b_5'$.

Let $e^* = a''b^* \in E(G)$ with $b^* \in V(B[b_1, b_2])$. Since G^* is 6-connected, we can choose e^* so that $b^* \notin \{b', b'', b_4\}$. Now $b^* \in B[b_4, b_2]$, to avoid the double cross e^*, e', e_4, e_5'. If $b^* \in B(b_4, b_5')$ then $(e_3, e_4, e^*, e_5', e')$ is a 5-edge configuration. If $b^* \in B(b_5', b')$ then $(e_3, e_4, e_5', e^*, e')$ is a 5-edge configuration. If $b^* \in B(b', b_2]$ then (e_3, e_4, e'', e', e^*) is a 5-edge configuration. $\square$

If $a_4 \neq a_2$ then there exist $e_i^* = a_i^* b_i^* \in E(G)$, $i \in [2]$, with $a_i^* \in A(a_4, a_2]$ and $b_i^* \in V(B(b_4, b_2])$, and we choose them so that $B[b_1^*, b_2^*]$ is maximal, and b_1, b_1^*, b_2^*, b_2 occur on B in order.

(8) If $a_4 \neq a_2$, then G has no edge from $B(b_1^*, b_2^*)$ to a_5.

We show that if (8) fails, then the desired 5-edge configuration exists, or splitting a_5 or b_5 results in a graph H such that (H, a_1, b_2, a_0, b_1) is planar, contradicting (i) of Lemma 2.5.1.

So assume $a_4 \neq a_2$ and that G has an edge e_5^* from $b_5^* \in B(b_1^*, b_2^*)$ to a_5. We see that $b_2^* \neq b_2$. For otherwise, $b_2^* = b_2$ and $a_2^* \neq a_2$. By (3), G has no edge from a_2 to $B[b_1, b_4]$, and so G has an edge from a_2 to $B(b_4, b_2)$, which together with e_4, e_2^*, e_5^* forms a double cross.

We may assume that G has no edge from $B(b_4, b_1^*)$ to $A[a_1, a_2] - a_4$. For otherwise, let $e = ab \in E(G)$ with $b \in V(B(b_4, b_1^*))$ and $a \in V(A[a_1, a_2] - a_4)$. Then by the definition of b_1^*, b_2^*, we have $a \notin A(a_4, a_2]$. Moreover, $a \neq a_1$ to avoid the double cross e, e_4, e_5^*, e_1^*. But then $a \in A(a_1, a_4)$, and so $(e_3, e_4, e, e_1^*, e_5^*)$ is a 5-edge configuration.

Hence, by (3) and (4), we may assume that G has no edge from $B[b_1, b_1^*)$ to $A(a_4, a_2]$ and G has no cross from $B[b_1, b_1^*)$ to $A[a_1, a_2]$.

We may also assume that G has no edge from $B(b_2^*, b_2]$ to $A[a_1, a_2]$. For, suppose G has an edge e from $b \in B(b_2^*, b_2]$ to $a \in A[a_1, a_2]$. Note $a \neq a_4$ to avoid the double cross e_4, e_1^*, e_5^*, e, and $a \notin A(a_4, a_2]$ by the definition of b_1^*, b_2^*. If $a = a_1$ then $(e, e_2^*, e_5^*, e_1^*, e_4)$ is a 5-edge configuration. If $a \in A(a_1, a_4)$ then $(e_3, e_4, e_5^*, e_2^*, e)$ is a 5-edge configuration.

Moreover, we may assume that $G - \{a_5, b_5^*\} - a_4b_1^*$ has no edge from $B[b_1^*, b_2^*]$ to $A[a_1, a_4]$. For, suppose there exists $e = ab \in E(G)$ with $e \neq a_4b_1^*$, $a \in V(A[a_1, a_4] - a_5)$, and $b \in V(B[b_1^*, b_2^*] - b_5^*)$. First, assume $b \in B(b_5^*, b_2^*]$. Then $a \in A[a_1, a_5)$ to avoid the double cross e_4, e_5^*, e, e_1^*. Hence $a = a_1$ by (3), and $(e_2^*, e, e_5^*, e_1^*, e_4)$ is a 5-edge configuration. So $b \in B[b_1^*, b_5^*)$. Then $a \in A(a_5, a_4]$ to avoid the double cross e_4, e_5^*, e, e_1^*. We may assume $b = b_1^*$; or else, $b \in B(b_1^*, b_5^*)$, and $(e_2^*, e_5^*, e, e_1^*, e_4)$ is a 5-edge configuration. Since $e \neq a_4b_1^*$, $a \in A(a_5, a_4)$. Let $e_0 = ab_0 \in E(G)$ with $b_0 \in V(B[b_1, b_2]) \setminus \{b_4, b_1^*, b_5^*\}$ (as $d_G(a) \geq 6$). By (3), $b_0 \notin B[b_1, b_4)$. Now $b_0 \notin B(b_1^*, b_2^*] - b_5^*$ as $b = b_1^*$, and $b_0 \notin B(b_2^*, b_2]$ as G has no edge from $B(b_2^*, b_2]$ to $A[a_1, a_2]$. So $b_0 \in B(b_4, b_1^*)$, and $(e_3, e_4, e_0, e_1^*, e_5^*)$ is a 5-edge configuration.

We may further assume that G has no cross from $A(a_4, a_2]$ to $B[b_1^*, b_5^*) \cup B(b_5^*, b_2^*]$. For, suppose G has a cross $e' = a'b', e'' = a''b''$ with $a', a'' \in A(a_4, a_2]$ and $b', b'' \in B[b_1^*, b_5^*) \cup B(b_5^*, b_2^*]$, such that a_1, a', a'', a_2 occur on A in order. Then $b' \in B[b_1^*, b_5^*)$ to avoid the double cross e_4, e_5^*, e', e'', and so $b'' \in B[b_1^*, b_5^*)$. Moreover, $a_2^* \in A[a'', a_2]$ to avoid the double cross e_4, e_5^*, e'', e_2^*. But now, $(e_2^*, e_5^*, e', e'', e_4)$ is a 5-edge configuration.

Let $e' = a'b', e'' = a''b'' \in E(G)$ with $b' \in V(B[b_1^*, b_5^*))$, $b'' \in V(B(b_5^*, b_2^*])$, and $a', a'' \in V(A(a_4, a_2])$, such that $B[b', b'']$ is minimal. Then there exists $e_0 = b_5^*a_0 \in E(G)$ with $a_0 \in V(A[a_1, a')) \cup V(A(a'', a_2]) \setminus \{a_5\}$; for otherwise, by (6) and above claims, we can split a_5 to obtain a graph H from G such that (H, a_1, b_2, a_0, b_1) is planar, contradicting

(i) of Lemma 2.5.1. In fact, $a_0 \in A[a_1, a')$ to avoid the double cross e_5^*, e_0, e'', e_4.

We may assume that G has no edge from a_5 to $B(b_4, b_2] - b_5^*$ (and, hence, $b_5 = b_5^*$). For, suppose $e = a_5 b \in E(G)$ with $b \in V(B(b_4, b_2] - b_5^*)$. If $b \in B(b_5^*, b_2]$ then $b \in B(b_5^*, b_2^*]$ by (6) and $a_0 \in A(a_5, a')$ to avoid the double cross e_0, e, e_4, e'; now (e_2^*, e, e_0, e', e_4) is a 5-edge configuration. We may thus assume $b \in B(b_4, b_5^*)$. Then $a_0 \in A[a_1, a_5)$ to avoid the double cross e, e_0, e_4, e'. Let $e_6 = a_5 b_6 \in E(G)$ with $b_6 \notin \{b_4, b', b_5^*\}$. Then $b_6 \notin B(b_5^*, b_2]$ to avoid the double cross e_6, e_0, e_4, e'. Moreover, $b_6 \notin B(b', b_5^*)$; or else, $(e_2^*, e_0, e_6, e', e_4)$ is a 5-edge configuration. By (6), $b_6 \notin B(b_4, b')$. So $b_6 \in B[b_1, b_4)$. But then $(e_2^*, e_0, e, e_4, e_6)$ is a 5-edge configuration.

Hence, by above claims, we can obtain a new graph H from G by splitting b_5^* such that (H, a_1, b_2, a_0, b_1) is planar, which contradicts (i) of Lemma 2.5.1. $\square$

We let $u_1, u_2 \in B[b_1, b_2]$, such that b_1, u_1, u_2, b_2 occur on B in order, G has an edge f_i from a_2 to u_i for $i \in [2]$, and subject to this, $B[u_1, u_2]$ is maximal. By $d_G(a_2) \geq 4$, $u_1 \neq u_2$.

(9) If $a_4 \neq a_2$, then G has an edge from a_2 to $B(b_5, b_2]$.

For, suppose $a_4 \neq a_2$ and G has no edge from a_2 to $B(b_5, b_2]$. By the choice of e_4, $u_1, u_2 \in B(b_4, b_5]$.

We may assume that G has no edge from $B(u_1, u_2)$ to $A[a_1, a_2)$. For, suppose there exists $ab \in E(G)$ with $b \in V(B(u_1, u_2))$ and $a \in V(A[a_1, a_2))$. Then $a \neq a_5$ by (8), and $a \in A(a_5, a_2)$ to avoid the double cross e, e_4, e_5, f_1. If $b_5 \neq b_2$ then (e_5, f_2, e, f_1, e_4) is a 5-edge configuration. So $b_5 = b_2$. Then $u_2 \neq b_5$ and (e_3, f_1, e, f_2, e_5) is a 5-edge configuration.

We may also assume that G has no cross from $A[a_1, a_2)$ to $B[b_1, u_1]$. For, suppose there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in A[a_1, a_2)$ and $b', b'' \in B[b_1, u_1]$, such that e', e'' form a cross, and a_1, a', a'', a_2 occur on A in order. If $b'' \in B[b_1, b_4)$ then by the choice of e_4, e_5, we have $a'' \in A[a_1, a_5]$ and $a' = a_1$; now e', e'', e_4, e_5 form a double cross, a contradiction. So $b'' \in B[b_4, u_1]$. Let f denote an edge from a_2 to $B(u_1, u_2)$. Then $a' \neq a_1$ to avoid the double cross e', f, e_4, e_5. Now (e_3, e'', e', f, e_5) is a 5-edge configuration.

By (i) of Lemma 2.5.1, (G, a_1, b_2, a_0, b_1) is not planar. So there exist $e' = a'b', e'' = a''b'' \in E(G)$ with $a', a'' \in V(A[a_1, a_2))$ and $b', b'' \in V(B[u_2, b_2])$, such that e', e'' are parallel, and a_1, a', a'', a_2 occur on A in order. Now $a' \in A[a_4, a_2)$ to avoid the double cross e', e'', e_4, f_1, and $b'' \in B[u_2, b_5]$ to avoid the double cross e_5, e'', e_4, f_1. We may assume $b_5 = b_2$; otherwise, (e_5, e'', e', f_1, e_4) is a 5-edge configuration. So $u_2 \neq b_5$. Now, let $e = a''b \in E(G)$ with $b \notin \{b', b'', b_5\}$. Then $b \notin B[b_1, u_1]$ to avoid the double cross e, e'', f_2, e'. We may assume $b \notin B[u_2, b')$; otherwise, (e_3, f_1, e, e', e'') is a 5-edge configuration. Since G has no edge from $B(u_1, u_2)$ to $A[a_1, a_2)$, $b \in B(b', b_5)$. But now, (e_3, f_1, e', e, e_5) is a 5-edge configuration. $\square$

(10) G has no edge from $B(b_5, b_2]$ to $A(a_1, a_4)$.

For, suppose there exists $e = ab \in E(G)$ with $b \in V(B(b_5, b_2])$ and $a \in V(A(a_1, a_4))$. We choose e so that $B[b, b_2]$ is minimal. By (3), $a \in A(a_5, a_4)$. By (5), G has no edge from $B(b_4, b_5)$ to $A(a_1, a_4) - a_5$. Moreover, since the degree of a in G is at least 6, then we let $e_0 = ab_0$ with $b_0 \in B[b_1, b_2]$ and $b_0 \notin \{b_4, b_5, b\}$. Now, by (3) and (5), and by the definition of b, we have $b_0 \in B(b_5, b)$.

G has no edge from $A(a_4, a_2]$ to $B[b_1, b)$. For, suppose there exists $e' = a'b' \in E(G)$ with $a' \in A(a_4, a_2]$ and $b' \in B[b_1, b)$. Then by (3), $b' \notin B[b_1, b_4]$. So $b' \in B(b_4, b)$. But then, e, e', e_4, e_5 form a double cross.

G has no edge from b_4 to $A(a_5, a_4)$ or no edge from a_4 to $B(b_4, b)$; otherwise, such two edges together with e_5, e form a double cross, a contradiction.

Now, we see that G has an edge e' from a_1 to $b' \in B(b_4, b_2]$; otherwise, since G has no edge from b_4 to $A(a_5, a_4)$ or no edge from a_4 to $B(b_4, b)$, then combined with (3), (4), (6), and (7), we can obtain a new graph H from G by splitting a_4 or b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 2.5.1.

We also see that G has no edge from a_1 to $B(b_5', b)$; otherwise, such an edge together with e_3, e_4, e_5', e forms a 5-edge configuration, a contradiction.

Hence, $b' \in B(b_4, b_5'] \cup B[b, b_2]$. We further choose e' so that $B[b', b_2]$ is maximal. Moreover, we let $e'' = a_1 b'' \in E(G)$ with $b'' \in B(b_4, b_5'] \cup B[b, b_2]$ so that $B[b'', b_2]$ is minimal.

Now, assume $b'' \in B(b_4, b_5']$. Then by the choice of e'', G has no edge from a_1 to $B[b, b_2]$. Moreover, G has no edge from $B[b_1, b_4)$ to $A(a_1, a_2]$; otherwise, by (3), such an edge must end in $A(a_1, a_5]$, which together with e', e_4, e_5 forms a double cross. Hence, G has an edge e_6 from a_4 to $b_6 \in B(b_4, b_5)$; or else, we can obtain a new graph H from G by splitting b_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 2.5.1. Now, G has no edge from b_4 to $A(a_1, a_4)$; or else, such an edge together with e_5, e', e_6 forms a double cross. So we may assume $a_2 \neq a_4$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 2.5.1. Then $u_2 \in B[b, b_2]$ (by (7) and (9)). Moreover, $b_6 \notin B(b', b_5]$; otherwise, (f_2, e, e_6, e', e_4) is a 5-edge configuration. So G has no edge from a_4 to $B(b', b_5]$. Therefore, we can obtain a new graph H from G by splitting a_4 as s, s', such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, a contradiction to (i) of Lemma 2.5.1.

So we may assume $b'' \in B[b, b_2]$. Now, $a_2 = a_4$; otherwise, $u_2 \in B[b, b_2]$ (by (7) and (9)) and $(f_2, e'', e_0, e_5, e_4)$ is a 5-edge configuration.

We also claim that G has an edge e_6 from $a_6 \in A(a_1, a_2)$ to $b_6 \in B[b_1, b_4]$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 2.5.1.

Then $b_6 \notin B[b_1, b_4)$; otherwise, $a_6 \in A(a_1, a_5]$, and (e, e'', e_5, e_4, e_6) is a 5-edge configuration. Hence, $b_6 = b_4$, and G has no edge from a_5 to $B[b_1, b_4)$, which further implies $b_5' \neq b_5$ (as the degree of a_5 in G is at least 6).

Now, we may assume $u_2 \notin B[b, b_2]$. For, suppose not. Then G has no edge from $\{a_1, a_2\}$ to $B(b_4, b_5)$; otherwise, such an edge together with f_2, e'', e_5, e_6 forms a 5-edge configuration. Moreover, $a_6 \notin A(a_5, a_2)$; otherwise, $(f_2, e'', e_0, e_5, e_6)$ is a 5-edge configuration. But now, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, a contradiction to (ii) of Lemma 2.5.1.

Since $u_2 \notin B[b, b_2]$, then G has no edge from a_2 to $B[b, b_2]$. By (7), G has no edge from a_2 to $B(b_5', b)$. By (3), G has no edge from a_2 to $B[b_1, b_4)$. Since the degree of a_2 in G is at least 4, then G has an edge e_2' from a_2 to $B(b_4, b_5')$. Now, $a_6 \notin A(a_5, a_2)$; otherwise, e_6, e_5, e, e_2' form a double cross. Moreover, $b' \notin B(b_4, b)$ to avoid the double cross e', e_2', e_6, e. Hence, combined with (6), we can obtain a new graph H from G by splitting a_2 as s, s', such that (H, a_1, b_2, a_0, b_1) is planar, a contradiction to (i) of Lemma 2.5.1. □

Now, by (3), (8), (9), and (10), we have

(11) G has no edge from $A(a_1, a_5) \cup A(a_4, a_2]$ to $B(b_4, b_5)$ and no edge from $B[b_1, b_4) \cup B(b_5, b_2]$ to $A(a_5, a_4)$.

We may assume that

(12) $G - \{a_5b_4, a_4b_5\}$ has no parallel edges from $A[a_5, a_4]$ to $B[b_4, b_5]$.

For, otherwise, let $e' = a'b', e'' = a''b'' \in E(G)$ be parallel with $a', a'' \in V(A[a_5, a_4])$ and $b', b'' \in V(B[b_4, b_5])$, such that a_1, a', a'', a_2 occur on A in order, $e' \neq a_5b_4$, and $e'' \neq a_4b_5$.

We may further assume $b' = b_4$ for any choice of e', e''. For, suppose $b' \neq b_4$. If $b'' \neq b_5$ then (e_3, e_4, e', e'', e_5) is a 5-edge configuration. So assume $b'' = b_5$. Then $a'' \neq a_4$. Since $d_G(a'') \geq 6$, there exists $e = a''b \in E(G)$ with $b \in V(B[b_1, b_2]) \setminus \{b_4, b', b_5\}$. By (11), $b \in B(b_4, b_5) - b'$. If $b \in B(b_4, b')$ then (e_3, e_4, e, e', e'') is a 5-edge configuration. If $b \in B(b', b_5)$ then (e_3, e_4, e', e, e_5) is a 5-edge configuration.

Thus, $G - a_4b_5$ has no parallel edges from $B(b_4, b_5]$ to $A[a_5, a_4]$. Now, since $e'' \neq a_4b_5$ and $d_G(a'') \geq 6$, then by (11), we may choose e'' so that $b'' \in B(b_4, b_5)$. Since $e' \neq a_5b_4$, $a' \in A(a_5, a_4)$. Moreover, since $d_G(a') \geq 6$, there exists $e = a'b \in E(G)$ with $b \in V(B[b_1, b_2]) \setminus \{b_4, b'', b_5\}$. By (11), $b \in B[b_4, b_5]$. If $b \in B(b_4, b'')$ then (e_3, e_4, e, e'', e_5) is a 5-edge configuration. So assume $b \in B(b'', b_5)$.

We may assume that G has no edge from a_2 to $B[b_5, b_2]$; otherwise, (f_2, e_5, e, e'', e') is a 5-edge configuration. Hence, $a_4 = a_2$ (by (9)). Moreover, G has no edge from a_1 to

$B(b_4, b_5)$, to avoid forming a double cross with e', e_5, e''. Therefore, since $G - a_4b_5$ has no parallel edges from $B(b_4, b_5]$ to $A[a_5, a_4]$, it follows from (3), (4), and (11) that there is no cross from $B[b_1, b_4]$ to A and no parallel edges from $B(b_4, b_2]$ to A. Now (G, a_1, b_2, a_0, b_1) is planar, contradicting (i) of Lemma 2.5.1. $\square$

If G has no edge from a_1 to $B(b_4, b_2]$ then by (3), (4), (11), and (12), we can split a_5, a_4 to s, s' and t, t', respectively, in G to obtain a graph H such that $(H, a_0, b_1, a_1, s, t, s', t', a_2, b_2)$ is planar, contradicting (iii) of Lemma 2.5.1. So let $e_0 = a_1b_0$ with $b_0 \in V(B(b_4, b_2])$. Choose e_0 with $B[b_0, b_2]$ maximal, and let $e'_0 = a_1b'_0 \in E(G)$ with $b'_0 \in B(b_4, b_2]$ so that $B[b'_0, b_2]$ is minimal.

(13) $a_4 = a_2$ implies $A(a_5, a_2) \neq \emptyset$.

For, suppose $a_4 = a_2$ and $A(a_5, a_2) = \emptyset$. Then there exists $e = ab \in E(G)$ with $b \in V(B[b_1, b_4])$ and $a \in V(A(a_1, a_5])$; or else, by (3), (4) and (6), $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1.

Suppose there exists $e' = a_2b' \in E(G)$ with $b' \in V(B(b_4, b_5))$. Then G has no edge from a_1 to $B(b_4, b_5)$, as such an edge would form a double cross with e, e', e_5. So $b_0 \in B[b_5, b_2]$. Now G has an edge e^* from a_2 to $B(b'_5, b_2]$; otherwise, by (3), (4) and (6), (G, a_1, b_2, a_0, b_1) is planar, contradicting (i) of Lemma 2.5.1. Hence, (e^*, e_0, e'_5, e', e) is a 5-edge configuration.

So assume that G has no edge from a_2 to $B(b_4, b_5)$. Then, since $d_G(a_2) \geq 4$, $u_2 \in B(b_5, b_2]$.

Assume $b_0 \in B(b_4, b_5)$. Then $b \notin B[b_1, b_4)$ to avoid the double cross e_0, e, e_4, e_5. Since $d_G(a_5) \geq 6$, then $b'_5 \neq b_5$, and there exists $e''_5 = a_5b''_5 \in E(G)$ with $b''_5 \in V(B(b'_5, b_5))$. By (6), $b_0 \in B(b_4, b'_5]$. We may assume that G has no edge from a_1 to $B[b_5, b_2]$; otherwise, such an edge together with f_2, e''_5, e_0, e forms a 5-edge configuration. Hence, by (3), (4) and (6), we can obtain a new graph H from G by splitting b_4 such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1.

Therefore, $b_0 \notin B(b_4, b_5)$ for any choice of b_0. Then G has an edge from $B[b_1, b_4)$ to $A(a_1, a_5]$; otherwise, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1. Hence, we may choose e so that $b \in B[b_1, b_4)$. If $b_0' \in B(b_5, b_2]$ or $b_5' \neq b_5$ then $(f_2, e_0', e_5', e_4, e)$ is a 5-edge configuration. So assume $b_0 = b_0' = b_5$. Then we can obtain a new graph H from G by splitting b_5 such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. □

(14) We may assume $a_4 \neq a_2$.

For, suppose $a_4 = a_2$. By (13), let $a_6 \in V(A(a_5, a_2))$. Since $d_G(a_6) \geq 6$, there exist distinct $e_6' = a_6 b_6', e_6'' = a_6 b_6'' \in E(G)$ with $b_6', b_6'' \in V(B) \setminus \{b_4, b_5\}$ such that $B[b_6', b_6'']$ is maximal. Without loss of generality, assume b_1, b_6', b_6'', b_2 occur on B in order. By (11), $b_6', b_6'' \in B(b_4, b_5)$.

Suppose there exists $e'' = b'' a'' \in E(G)$ with $b'' \in V(B[b_1, b_4])$ and $a'' \in V(A(a_1, a_5])$. Then $b_0 \notin B(b_4, b_6']$ to avoid the double cross e_0, e'', e_5, e_6''. We may assume $b_0 \notin B(b_6', b_5)$; otherwise, $(e_3, e_4, e_6', e_0, e_5)$ is a 5-edge configuration. Hence, $b_0 \in B[b_5, b_2]$ and G has no edge from a_1 to $B(b_4, b_5)$. We also see that G has no edge from a_1 to $B(b_5, b_2]$ or no edge from a_2 to $B(b_5, b_2]$; otherwise, such two edges form a 5-edge configuration with e_5, e_6', e''. By (3), (4), (11), and (12), we can obtain a graph H from G by splitting a_2 such that (H, a_1, b_2, a_0, b_1) is planar, contradicting (i) of Lemma 2.5.1.

Thus, we may assume that G has no edge from $B[b_1, b_4]$ to $A(a_1, a_5]$. Hence, by (11) and (12), $(G - a_1, a_2, b_2, a_0, b_1)$ is planar. Now, by (ii) of Lemma 2.5.1, $(G - a_2, a_1, b_2, a_0, b_1)$ is not planar; hence, there exist $e = a_1 b, e' = a' b' \in E(G)$ with $b \in V(B(b_4, b_5))$, $b' \in V(B[b_1, b))$, and $a' \in V(A(a_1, a_2))$. We may assume $b \notin B(b_6', b_5)$, as otherwise (e_3, e_4, e_6', e, e_5) is a 5-edge configuration. Moreover, G has no edge from a_2 to $B(b_4, b_5)$, as such an edge would form a double cross with e, e', e_5. Since $d_G(a_2) \geq 4$, $u_2 \in B[b_5, b_2]$. But now, (f_2, e_5, e_6'', e, e') is a 5-edge configuration. □

Now, by (9) and (14), $u_2 \in B(b_5, b_2]$. By (3), (11) and (14), G has no edge from a_2 to $B[b_1, b_5)$, and so $u_1 \in B[b_5, b_2]$.

(15) $b_0 \in B(b_4, b_5)$.

For, otherwise, $b_0 \in B[b_5, b_2]$. Note that $b_0' \neq b_5$; otherwise, $b_0 = b_0' = b_5$, and by (3), (4), (11), (12), and (14), we can obtain a new graph H from G by splitting a_4 such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1.

We may assume that G has no edge from $B[b_1, b_4)$ to $A(a_1, a_5]$, as such an edge forms a 5-edge configuration with f_2, e_0', e_5, e_4. Hence, $A(a_1, a_5) = \emptyset$ and, since $d_G(a_5) \geq 6$, $b_5' \neq b_5$. We may thus assume that G has no edge from $B[b_5, b_0')$ to $A[a_4, a_2)$, as such an edge forms a 5-edge configuration with f_2, e_0', e_5', e_4. We may also assume that if $b_4a_5 \in E(G)$ then G has no edge from $B(b_4, b_5)$ to $A(a_5, a_2]$, as such an edge forms a 5-edge configuration with f_2, e_0', e_5, b_4a_5.

Suppose $u_1 \notin B[b_5, b_0')$. Then by definition, G has no edge from $B[b_5, b_0')$ to $A[a_4, a_2]$. Now, by (3), (4), (11), (12), and our previous statements, we can obtain a new graph H from G by splitting a_1, a_4 as s, s' and t, t', respectively, such that $(H, a_0, b_1, a_1 = s, t, s', t', a_2, b_2)$ is planar, contradicting (iii) of Lemma 2.5.1.

So $u_1 \in B[b_5, b_0')$ and, hence, G has no edge from $B[b_5, b_2]$ to $A[a_4, a_2)$. By (3), (4), (11), (12), and our previous statements, $(G - a_1, a_2, b_2, a_0, b_1)$ and $(G - a_2, a_1, b_2, a_0, b_1)$ are planar, contradicting (ii) of Lemma 2.5.1. $\square$

Suppose there exists $a \in V(A(a_5, a_4))$. Since $d_G(a) \geq 6$ and because of (11), there exists $e = ab \in E(G)$ with $b \in V(B[b_4, b_5]) \setminus \{b_4, b_5, b_0\}$. If $b \in B(b_4, b_0)$ then (e_3, e_4, e, e_0, e_5) is a 5-edge configuration; if $b \in B(b_0, b_5)$ then (f_2, e_5, e, e_0, e_4) is a 5-edge configuration.

So we may assume $A(a_5, a_4) = \emptyset$. Then G has no edge from $A(a_1, a_5]$ to $B[b_1, b_4)$, as such an edge would form a double cross with e_0, e_4, e_5.

Then we may assume that G has no edge from $B(b_0, b_0')$ to $A(a_1, a_2]$. For, suppose there exists $e = ab \in E(G)$ with $b \in V(B(b_0, b_0'))$ and $a \in V(A(a_1, a_2])$. If $b_0' \in B(b_5, b_2]$, then $(f_2, e_0', e_5, e_0, e_4)$ is a 5-edge configuration. So assume $b_0' \in B(b_4, b_5]$. Then $b \in B(b_4, b_5)$ and, by (11), $a \in A[a_5, a_4]$. But then, (f_2, e_0', e, e_0, e_4) is a 5-edge configuration.

If G has no edge from a_4 to $B(b_4, b_5)$ then, by (3), (4), (6), (11), and our previous statements, we can obtain a new graph H from G by splitting b_4 such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. So let $e = a_4 b \in E(G)$ with $b \in V(B(b_4, b_5))$. We may assume $b \notin B(b_0, b_5)$; otherwise (f_2, e_5, e, e_0, e_4) is a 5-edge configuration. Moreover, G has no edge from b_4 to a_5, to avoid forming a double cross with e_5, e_0, e. Now by (3), (4), (6), (11), and our previous statements, we can obtain a new graph H from G by splitting a_4 such that $(H, a_1, a_2, b_2, a_0, b_1)$ is planar, contradicting (i) of Lemma 2.5.1. $\square$

Lemma 2.5.3 *Suppose $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration in an ideal a_0-frame A, B in γ with $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ on B in order. Let $G_0 := G - A$, where (G_0, a_0, b_1, B, b_2) is planar. Then G_0 has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 3$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, and $B[b_1', b_2'] \subseteq G_2$, $|V(G_1 - G_2)| \geq 1$, such that one of the following holds for $b_1', b_2' \in V(G_1) \cap V(G_2)$:*

(i) $|V(G_1) \cap V(G_2)| = 3$, $b_1' \in B[b_3, b_4]$, $b_2' \in B[b_7, b_2]$, and G_0 has a path from a_0 to $B(b_1', b_2')$ and internally disjoint from B.

(ii) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_3, b_4]$, and $b_2' \in B[b_7, b_2]$.

(iii) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_3, b_4]$, and $b_2' \in B[b_6, b_7)$.

(iv) $|V(G_1) \cap V(G_2) = 2$, $b_1' \in B(b_4, b_5]$, and $b_2' \in B[b_7, b_2]$.

Proof. By planarity of G_0, it is easy to see that if the assertion fails then $G_0 - (B[b_3, b_4] \cup B[b_7, b_2])$ contains disjoint paths B_1, A_0 from b_1, a_0 to b_5, b_6, respectively. Now $(A - A[a_5, a_7]) \cup e_3 \cup B[b_3, b_4] \cup e_4 \cup e_6 \cup A_0$ and $B_1 \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B[b_7, b_2]$ show that γ is feasible, a contradiction. (See Figure 2.13.) $\square$

In the remainder of this section, we will assume the following: $\mathcal{P} := (e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration in A, B, where $e_i = a_i b_i \in E(G)$ with $a_i \in V(A)$ and $b_i \in V(B)$ for $i = 3, 4, 5, 6, 7$, such that a_1, a_3, a_4, a_2 occur on A in order, $b_1, b_3, b_4, b_5, b_6, b_7, b_2$ occur on B in order, and the following are satisfied in order listed:

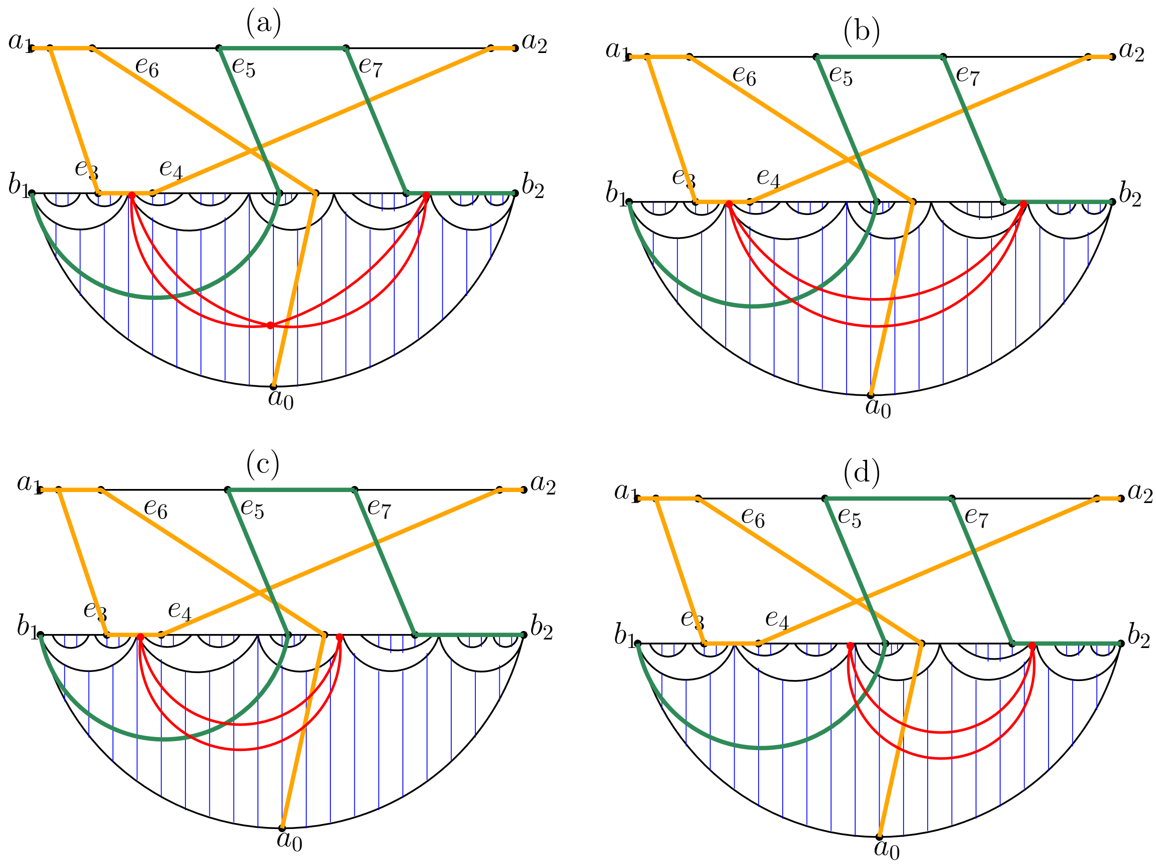

Figure 2.13: A 5-edge configuration with a 2-cut or a 3-cut

- $B[b_4, b_7]$ is maximal,
- $B[b_6, b_7]$ is minimal,
- $B[b_4, b_5]$ is minimal,
- $A[a_5, a_7]$ is minimal,
- $A[a_3, a_4]$ is maximal,
- $B[b_1, b_3]$ is minimal, and
- $A[a_6, a_5] \cap A[a_6, a_7]$ is maximal.

Lemma 2.5.4 *Suppose* $a_7 \in A[a_1, a_5], a_6 \in A(a_5, a_2]$*, and* G *has no edge from* $B(b_4, b_5]$ *to* $A[a_1, a_5)$ *or from* $B[b_7, b_2]$ *to* $A(a_5, a_2]$*. Then* G_0 *admits no separation* (G_1, G_2) *such that*

$V(G_1 \cap G_2) = \{b_1^*, b_2^*\}$ *with* $b_1^* \in V(B[b_1, b_4])$ *and* $b_2^* \in V(B[b_6, b_2])$, $\{a_0, b_1, b_2\} \subseteq V(G_1)$, $B[b_1^*, b_2^*] \subseteq G_2$, *and* $|V(G_1 - G_2)| \geq 1$.

Proof. For, suppose such a separation does exist. Then we choose such (G_1, G_2) so that $B[b_1^*, b_2^*]$ is maximal. Note that G has no parallel edges from $B[b_6, b_2]$ to $A[a_1, a_5]$, as such edges and e_5, e_6 would form a double cross.

Next, we show that all edges from $A(a_5, a_2]$ to B must end in $B[b_4, b_6]$. For, suppose there exists $e = ab \in E(G)$ with $a \in V(A(a_5, a_2])$ and $b \in V(B) \setminus V(B[b_4, b_6])$. Then $b \in B[b_1, b_4)$; for, otherwise, $b \in B(b_6, b_7)$ (as G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$) and, hence, (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$. If $a \in A(a_5, a_4)$ then $b \in B[b_3, b_4)$ to avoid the double cross e, e_3, e_4, e_5; thus $b_3 \neq b_4$ and (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Hence, $a \in A[a_4, a_2]$. Then $b = b_1$ as, otherwise, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Now $a \neq a_2$ and there exists $e' = a_2 b' \in E(G)$ with $b' \in V(B) - \{b_1, b_2\}$. Note that $b' \notin B[b_7, b_2]$ as G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. But then e, e', e_3, e_7 form a double cross, a contradiction.

Let $e_8 = a_8 b_8 \in E(G)$ with $a_8 \in V(A[a_1, a_5])$ and $b_8 \in V(B(b_1^*, b_2^*))$, so that $A[a_1, a_8]$ is minimal. Since G^* is 6-connected, there exists $e^* = a^* b^* \in E(G)$ with $a^* \in A(a_8, a_2]$ and $b^* \in B - B[b_1^*, b_2^*]$. Since all edges from $A(a_5, a_2]$ to B end in $B[b_4, b_6]$, $a^* \in A(a_8, a_5]$ and, hence, $a_8 \in A[a_1, a_5)$.

Moreover, $b_8 \in B(b_1^*, b_4] \cup B[b_6, b_2^*)$. For otherwise, $b_8 \in B(b_4, b_6)$. Since $a_8 \in A[a_1, a_5)$ and G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$ (by assumption), $b_8 \in B(b_5, b_6)$. Then $a_8 \in A[a_7, a_5)$ to avoid the double cross e_5, e_6, e_7, e_8. Since $a^* \in A(a_8, a_5]$, we have $b^* \in B[b_1, b_1^*)$ to avoid the double cross e_8, e^*, e_5, e_6, and $b^* \notin B[b_1, b_3)$ to avoid the double cross e_3, e^*, e_6, e_7. Hence, $b_3, b^* \in B(b_1, b_4)$, and $(e_3, e^*, e_8, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Case 1. $b_8 \in B[b_6, b_2^*)$. So $b^* \in B[b_1, b_1^*)$ to avoid the double cross e_8, e^*, e_5, e_6.

We claim that G has no edge from $B(b_1^*, b_4]$ to $A[a_1, a_5)$. For suppose $e = ab \in E(G)$ with $a \in A[a_1, a_5)$ and $b \in B[b_1^*, b_4]$. Note that b_1^* and b_2^* are feet of some connector J, and

$B[b_1^*, b_2^*] \subseteq J$. Let u_1, u_2 denote the extreme hands for J. Note that e^* is from $A(x_1, x_2)$ to $B[b_1, b_1^*)$; so we know $(J - b_1^*, u_1, A(u_1, u_2), u_2, b_2^*)$ is planar by Lemma 2.2.4. But this cannot be the case because of e, e_4, e_5.

Let (G_1', G_2') be a separation in G_0 such that $V(G_1' \cap G_2') = \{b_1', b_2'\}$ with $b_1^*, b_1', b_4, b_6, b_2', b_2^*$ on B in order, $B[b_1', b_2'] \subseteq G_1'$, and $\{a_0, b_1', b_2'\} \subseteq V(G_2')$. (Possibly $G_i' = G_i$ for $i = 1, 2$.) We choose (G_1', G_2') such that $B[b_6, b_2']$ is minimal and, subject to this, $B[b_1^*, b_1']$ is minimal.

Let $e_8' = a_8'b_8' \in E(G)$ with $a_8' \in A[a_1, a_5]$ and $b_8' \in B(b_1', b_2')$, and choose e_8' so that $A[a_1, a_8']$ is minimal. Since G^* is 6-connected, there exists $e' = a'b' \in E(G)$ with $a' \in A(a_8', a_2]$ and $b' \in B - B[b_1', b_2']$. Then $b_8' \in B[b_6, b_2')$ (by the claim above) and $b' \in B[b_1, b_1^*] - b_1'$ (to avoid the double cross e_5, e_6, e_8', e'). So $(e_8', e_6, e_5, e_4, e')$ is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3, G_0 has a cut that contradicts the choice of (G_1, G_2) or (G_1', G_2').

Case 2. $b_8 \in B(b_1^*, b_4]$. Then $b^* \in B(b_2^*, b_2]$ to avoid the double cross e_8, e^*, e_4, e_5.

We claim that G has no edge from $B[b_6, b_2^*)$ to $A[a_1, a_5)$. For suppose $e = ab \in E(G)$ with $a \in V(A[a_1, a_5))$ and $b \in V(B[b_6, b_2^*))$. Note that b_1^* and b_2^* are feet of some connector J, and $B[b_1^*, b_2^*] \subseteq J$. Let u_1, u_2 denote the extreme hands for J. Note that e^* is from $A(u_1, u_2)$ to $B(b_2^*, b_2]$; so we know $(J - b_2^*, u_1, A(u_1, u_2), u_2, b_1^*)$ is planar by Lemma 2.2.4. But this cannot be the case because of e, e_5, e_6.

Let (G_1', G_2') be a separation in G_0 such that $V(G_1' \cap G_2') = \{b_1', b_2'\}$ with $b_1^*, b_1', b_4, b_6, b_2', b_2^*$ on B in order, $B[b_1', b_2'] \subseteq G_1'$, and $\{a_0, b_1', b_2'\} \subseteq V(G_2')$. We choose (G_1', G_2') such that $B[b_1', b_4]$ is minimal and, subject to this, $B[b_2', b_2^*]$ is minimal.

Let $e_8' = a_8'b_8' \in E(G)$ with $a_8' \in A[a_1, a_5]$ and $b_8' \in B(b_1', b_2')$, and choose e_8' so that $A[a_1, a_8']$ is minimal. Since G^* is 6-connected, there exists $e' = a'b' \in E(G)$ with $a' \in A(a_8', a_2]$ and $b' \in B - B[b_1', b_2']$. Then $b_8' \in B(b_1', b_4]$ (by the above claim) and $b' \in B[b_2^*, b_2] - b_2'$ (to avoid the double cross e', e_8', e_4, e_5). So $(e_8', e_4, e_5, e_6, e')$ is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3, G_0 has a separation that contradicts choice of

(G_1, G_2) or (G'_1, G'_2). $\square$

Lemma 2.5.5 *Suppose G_0 has a 2-cut $\{b'_1, b'_2\}$ with $b'_1 \in B[b_1, b_4]$ and $b'_2 \in B[b_6, b_7)$ separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$. Then G_0 has a separation (G_1, G_2) with $|V(G_1 \cap G_2)| \leq 3$ and $b^*_1, b^*_2 \in V(G_1 \cap G_2) \cap V(B)$ such that $b^*_1 \in B[b_1, b_4]$, $b^*_2 \in B[b_6, b_2]$, $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $B[b^*_1, b^*_2] \subseteq G_2$, and if $b^*_2 \in B[b_6, b_7)$ then $|V(G_1 \cap G_2)| = 2$ and G has no edge from $B(b^*_2, b_7)$ to $A - a_7$.*

Proof. We choose $\{b'_1, b'_2\}$ such that $\{b'_1, b'_2\}$ is a 2-cut (with $b'_1 \in B[b_1, b_4]$ and $b'_2 \in B[b_6, b_2]$) separating $B[b'_1, b'_2]$ from $\{a_0, b_1, b_2\}$, subject to this, $B[b'_1, b_4]$ is minimal and, subject to this, $B[b'_2, b_2]$ is minimal.

Clearly, we may assume $b'_2 \in B[b_6, b_7)$, and there exists $e_8 = a_8b_8 \in E(G)$ with $a_8 \in V(A - a_7)$ and $b_8 \in V(B(b'_2, b_7))$. We choose e_8 so that $A[a_8, a_5]$ is minimal. Note that $a_8 \in A[a_5, a_7)$, for otherwise, $(e_3, e_4, e_5, e_8, e_7)$ contradicts $\mathcal{P}$.

Case 1. $a_5 \in A(a_7, a_2]$.

Then G has no edge from $A(a_8, a_5]$ to $B[b_1, b_3)$ to avoid forming a double cross with e_3, e_8, e_4. Also G as no edge from $A(a_5, a_2]$ to $B(b_1, b'_1)$; for suppose e is such an edge then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$.

(1) G has no edge from $A(a_8, a_2]$ to $B(b'_2, b_2) + b_1$.

For, suppose there exists $e = ab \in E(G)$ with $a \in A(a_8, a_2]$ and $b \in B(b'_2, b_2) + b_1$. If $b = b_1$ then $a \neq a_2$ and there exists $e_2 = a_2b' \in E(G)$ with $b' \in B(b_1, b_2)$; now $b' \in B[b_7, b_2)$ to avoid the double cross e, e_3, e_7, e_2 and, hence, (e_2, e_7, e_5, e_3, e) contradicts the choice of $\mathcal{P}$.

Thus, $b \in B(b'_2, b_2)$. In fact $b \in B[b_7, b_2)$, otherwise, $a \in A(a_5, a_2]$ (by the minimality of $A[a_8, a_5]$) and (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$. Now $a \in A(a_5, a_2]$, as otherwise (e_3, e_4, e_5, e_6, e) contradicts the choice of $\mathcal{P}$. Hence, (e, e_7, e_8, e_6, e_5) is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b'_1, b'_2\}$, G_0 has the desired separation. $\square$

(2) G has no edge from $A(a_7, a_2]$ to b_2.

For, let $e = ab_2 \in E(G)$ with $a \in A(a_7, a_2]$. Then $a \neq a_2$. Moreover, $a \in A(a_5, a_2)$; as otherwise, (e_3, e_4, e_5, e_6, e) contradicts the choice of $\mathcal{P}$.

Suppose $a \in A[a_4, a_2)$. Then let $e_2 = a_2b_2' \in E(G)$ with $b_2' \in V(B) - \{b_1, b_2\}$. Now $b_2' \in B(b_1, b_4]$ to avoid the double cross e_2, e, e_4, e_8. So $(e_3, e_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Thus $a \in A(a_5, a_4)$. Now $b_7 = b_2$, or else (e_3, e_4, e_5, e_7, e) contradicts the choice of $\mathcal{P}$. Moreover, $a_8 = a_5$, or else (e_3, e_4, e_5, e_8, e) contradicts the choice of $\mathcal{P}$.

Suppose $a_6 \in A[a_1, a_7)$. Let $e_7' = a_7b_7' \in E(G)$ with $b_7' \in V(B - b_7)$. Then $b_7' \notin B[b_1, b_6)$ to avoid the double cross e_6, e_7', e_7, e_8. If $b_7' = b_6$ then $(e_3, e_4, e_7', e_8, e_7)$ contradicts the choice of $\mathcal{P}$. If $b_7' \in B(b_6, b_2)$ then $(e_3, e_4, e_5, e_7', e_7)$ contradicts the choice of $\mathcal{P}$.

So $a_6 \in A(a_5, a_2]$ for all choices of e_6. Then $a_6 \in A[a_4, a_2]$, or else (e_3, e_4, e_6, e_8, e) contradicts the choice of $\mathcal{P}$. Let $e' = ab' \in E(G)$ with $b' \in V(B - b_2)$. Then $b' \neq b_6$ as $a_6 \in A[a_4, a_2]$ for all choices of e_6. So $b' \in B(b_6, b_2)$ to avoid the double cross e_8, e_6, e, e'. But then (e_3, e_4, e_5, e', e_7) contradicts the choice of $\mathcal{P}$. $\square$

(3) There exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in A[a_1, a_8)$ and $b_9 \in B(b_1', b_2']$.

For, suppose such an edge does not exist. Then $a_6 \in A(a_5, a_2]$ and G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$ by the choice of $\mathcal{P}$. Note that we have $a_5 \neq a_7$ and $a_7 \in A[a_1, a_5]$ and that, by (1) and (2), G has no edge from $B[b_7, b_2]$ to $A(a_5, a_2]$. This contradicts Lemma 2.5.4. $\square$

(4) $b_9 \in B(b_4, b_2']$ and $a_9 = a_3$; so all edges from $B(b_1', b_2']$ to $A[a_1, a_8)$ must be from $B(b_4, b_2']$ to a_3.

First, suppose $b_9 \in B(b_1', b_4]$. Then $(e_9, e_4, e_5, e_6, e_8)$ is 5-edge configuration. Thus, by Lemma 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

So we may assume $b_9 \in B(b_4, b_2']$. Suppose $a_9 \neq a_3$. Then $a_9 \in A(a_3, a_4)$, to avoid the double cross e_3, e_9, e_5, e_7. But now $(e_3, e_4, e_9, e_8, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. $\square$

Suppose $a_4 \neq a_2$. Let $e_2^* = a_2 b_2^* \in E(G)$ with $b_2^* \in V(B)$. Then $b_2^* \in B(b_1, b_4]$ to avoid the double cross e_2^*, e_4, e_9, e_8. Now (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Thus, G has no edge e from $B[b_1, b_1')$ to $v \in V(A(a_8, a_2])$; for, if $v \neq a_2$ then e, e_9, e_8, e_4 would form a double cross, and if $v = a_2$ then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Hence, by (1) and (4), G has a 5-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b_1', b_2', a_8, a_3, a_2\}$, $V(A[a_8, a_2]) \cup V(B[b_1', b_2']) \cup \{a_3\} \subseteq V(H_1)$, and $V(A[a_3, a_8]) \cup \{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_2)$, a contradiction as G^* is 6-connected.

Case 2. $a_5 \in A[a_1, a_7)$.

Then $a_6 \notin A(a_4, a_2)$ to avoid the double cross e_4, e_6, e_5, e_7, and $a_6 \notin A(a_7, a_4)$ as, otherwise, $(e_3, e_4, e_6, e_8, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $a_6 \in A[a_1, a_5)$ or $a_6 = a_4$.

(1) For some $v \in \{a_4, b_4\}$, all edges from $A(a_8, a_2]$ to $B(b_1', b_2']$ are incident with v.

To prove (1), we first claim that G has no edge from $A(a_8, a_2] - a_4$ to $B(b_1', b_2'] - b_4$. For otherwise, suppose there exists $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in A(a_8, a_2] - a_4$ to $b_9 \in B(b_1', b_2'] - b_4$. If $b_9 \in B(b_1', b_4)$ then $a_9 \in A(a_4, a_2]$ to avoid the doublecorss e_9, e_4, e_7, e_8; so $(e_3, e_9, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B(b_4, b_2')$. Then $a_9 \in A(a_8, a_4)$ to avoid the double cross e_4, e_9, e_8, e_7. Now $(e_3, e_4, e_9, e_8, e_7)$ contradicts the choice of $\mathcal{P}$.

Next, observe that, by the choice of $\mathcal{P}$, any edge from b_4 to $A(a_8, a_2] - a_4$ must end in $A(a_8, a_4)$, and any edge from a_4 to $B(b_1', b_2'] - b_4$ must end in $B(b_4, b_2']$. Thus, G has no edge from b_4 to $A(a_8, a_2] - a_4$ or no edge from a_4 to $B(b_1', b_2'] - b_4$; as such two edges and e_7, e_8 would form a double cross, a contradiction. $\square$

Define $a_1' \in V(A[a_1, a_8])$ such that G has no edge from $A[a_1, a_1')$ to $B(b_1', b_2']$ and, subject to this, $A[a_1, a_1']$ is maximal. By the definition of a_1', there exits $e_1 = a_1' b \in E(G)$ with $b \in B(b_1', b_2']$.

We claim that $a_1' \in A[a_3, a_8)$. For, suppose $a_1' \in A[a_1, a_3)$. Then $b \in B(b_1, b_3]$ to avoid the double cross e_1, e_3, e_4, e_8. Now $(e_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

(2) G has no edge from $A(a_1', a_8)$ to $B - B[b_1', b_2']$.

For, otherwise, $a_1' \neq a_8$, and there exists $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A(a_1', a_8))$ to $b_9 \in V(B) \setminus V(B[b_1', b_2'])$. Then $b_9 \notin B[b_1, b_1')$ to avoid the double cross e_1, e_9, e_4, e_7.

We claim $b_9 = b_2$ and $a_9 \notin A[a_5, a_8)$. For, if $b_9 \in B(b_2', b_7)$ then $a_9 \in A(a_1', a_5)$ by the choice of e_8 (that $A[a_5, a_8]$ is minimal); now $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$. Hence, $b_9 \in B[b_7, b_2]$. Thus, $a_9 \notin A[a_5, a_8)$; as otherwise $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $\mathcal{P}$. Now suppose $b_9 \neq b_2$. Then $(e_7, e_9, e_8, e_6, e_5)$ is a 5-edge configuration. Thus, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

Now $a_8 = a_5$; otherwise, $(e_3, e_4, e_5, e_8, e_9)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_4 = a_2$; for otherwise, G has an edge e' from a_2 to B, then either (e_3, e', e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$ or e', e_4, e_5, e_7 form a double cross.

Next, we claim that all edges from $A(a_8, a_2)$ to B must end in $\{b_4, b_2\}$. Note that G has no edge from $A(a_8, a_2)$ to b_1, to avoid forming a double cross with e_7, e_3, e_4. G has no edge from $A(a_8, a_2)$ to $B(b_1, b_4)$; otherwise, such an edge together with e_3, e_5, e_1, e_9 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. G has no edge from $A(a_8, a_2)$ to $B(b_4, b_8)$; otherwise, such an edge together with e_3, e_4, e_8, e_7 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$. G has no edge from $A(a_8, a_2)$ to $B[b_8, b_2)$; otherwise, such an edge together with e_3, e_4, e_5, e_9 forms a 5-edge configuration contradicting the choice of $\mathcal{P}$.

Therefore, since $a_7 \in A(a_8, a_2)$, $\{a_2, a_8, b_2, b_4\}$ is a 4-cut in G separating a_7 from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction as G^* is 6-connected. $\square$

By (1) and (2), G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{b_1', b_2', a_8, a_1', v\}$, $b_5 \in V(H_2 - H_1)$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq H_1$, a contradiction as G^* is 6-connected. $\square$

Lemma 2.5.6 *Suppose G_0 has a 2-cut $\{b_1', b_2'\}$ separating $B[b_1', b_2']$ from $\{a_0, a_1, a_2\}$ with $b_1' \in B(b_4, b_5]$ and $b_2' \in B[b_7, b_2]$. Then G_0 has a separation (G_1, G_2) with $|V(G_1 \cap G_2)| \leq 3$ and $b_1^*, b_2^* \in V(G_1 \cap G_2) \cap V(B)$ such that $b_1^* \in B[b_1, b_5]$, $b_2^* \in B[b_7, b_2]$, $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $B[b_1^*, b_2^*] \subseteq G_2$, and if $b_1^* \in B(b_4, b_5]$ then $|V(G_1 \cap G_2)| = 2$ and G has no edge from $B(b_4, b_1^*)$ to $A - a_4$.*

Proof. We choose $\{b_1', b_2'\}$ such that $\{b_1', b_2'\}$ is a 2-cut (with $b_1' \in B[b_1, b_5]$ and $b_2' \in B[b_7, b_2]$) separating $B[b_1', b_2']$ from $\{a_0, b_1, b_2\}$, and, subject to this, $B[b_1', b_2']$ is maximal. Clearly, we may assume $b_1' \in B(b_4, b_5]$, and there exists $e_8 = a_8 b_8 \in E(G)$ with $a_8 \in V(A - a_4)$ and $b_8 \in V(B(b_4, b_1'))$.

We claim that $a_8 \in A[a_1, a_3] \cup A(a_4, a_2]$. For, suppose $a_8 \in A(a_3, a_4)$. Then $a_6 \in A[a_7, a_8]$ and $a_8 \notin A[a_7, a_5]$; for otherwise $(e_3, e_4, e_8, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_5 \notin A[a_6, a_8]$ (since $a_6 \notin A[a_5, a_7]$). So $(e_3, e_4, e_8, e_5, e_6)$ is a 5-edge configuration. Thus, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

Case 1. $a_8 \in A(a_4, a_2]$.

Choose e_8 so that $A[a_8, a_2]$ is minimal. Note that $a_6 \in A[a_8, a_2]$ and $a_7 \in A(a_3, a_5]$, since, otherwise, e_4, e_8 and two of $\{e_5, e_6, e_7\}$ force a double cross.

(1) G has no edge from $A(a_5, a_2]$ to $B[b_1, b_4) \cup B(b_6, b_2]$.

For, let $e = ab \in E(G)$ with $a \in A(a_5, a_2]$ and $b \in B[b_1, b_4) \cup B(b_6, b_2]$.

Suppose $b \in B(b_6, b_2]$. Then $a \in A[a_8, a_2]$ to avoid the double cross e, e_4, e_5, e_8. So $b \in B[b_7, b_2]$, or else (e_3, e_4, e_5, e, e_7) contradicts the choice of $\mathcal{P}$. If $b = b_2$ then $a \neq a_2$ and there exists $e' = a_2 b' \in E(G)$ with $b' \in V(B(b_1, b_2))$; e_4, e_5, e, e' form a double cross (when $b' \in B(b_4, b_2)$) or (e_3, e', e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$ (when $b' \in B(b_1, b_4]$). Thus, $b \neq b_2$. Now (e, e_7, e_5, e_8, e_4) is a 5-edge configuration. Hence, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

Thus, $b \in B[b_1, b_4)$ for every choice of $e = ab$. If $a \in A(a_5, a_4)$ then either e_3, e_4, e_5, e form a double cross, or (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. So $a \in A[a_4, a_2]$. Then $b = b_1$, or else, (e, e_3, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. Now, since G has no edge from $B(b_6, b_2]$ to $A(a_5, a_2]$, G has an edge from a_2 to $B(b_1, b_7)$, which forms a double cross with e, e_3, e_7. $\square$

(2) G has no edge from $B(b_1, b_3)$ to A.

For otherwise, let $e = ab \in E(G)$ with $a \in A$ and $b \in B(b_1, b_3)$. If $a \in A[a_1, a_3]$, then (e, e_4, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$; if $a \in A(a_3, a_4)$, then e, e_3, e_4, e_7 form a double cross; if $a \in A[a_4, a_2]$, then (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. $\square$

(3) $b_2' = b_2$ and G_0 has a separation (G_1', G_2') that $V(G_1' \cap G_2') = \{b_1, b_2'', a_0\}$, $b_2'' \in B(b_1', b_2')$, $B[b_1, b_2''] \subseteq G_1'$, and $\{a_0, b_1, b_2\} \subseteq V(G_2')$.

First, suppose $b_5 \in B(b_1', b_2')$ and there exist $e_5' = a_5 b_5', e_5'' = a_5' b_5 \in E(G)$ with $a_5' \in A[a_1, a_8)$ and $b_5' \in B(b_1', b_2')$ such that $a_5' \neq a_5$ and $b_5' \neq b_5$. Then e_5', e_5'' form a cross to avoid the double cross e_5', e_5'', e_4, e_8. Hence, $b_5' \in B(b_5, b_2')$ by the choice of $\mathcal{P}$, and so $(e_6, e_5', e_5'', e_8, e_4)$ is a 5-edge configuration. By Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, we see that (3) or the assertion of the lemma holds.

So the above case will not happen. Then we claim that there exists $v \in \{a_5, b_5\}$ such that all edges from $B(b_1', b_2')$ to $A[a_1, a_8)$ in G contain v. For, otherwise, there exists $e = ab \in E(G)$ such that $a \in V(A[a_1, a_8) - a_5)$ and $b \in V(B(b_1', b_2') - b_5)$. Suppose $b \in B(b_1', b_5)$. Then $a \in A(a_5, a_8)$ to avoid the double cross e, e_5, e_4, e_8, and, hence, (e_6, e_5, e, e_8, e_4) is a 5-edge configuration. Now by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1' b_2'\}$, (3) or the assertion of the lemma holds. So assume $b \in B(b_5, b_2')$. Then $a \notin A(a_5, a_8)$ to avoid the double cross e_4, e_5, e_8, e. Hence, (e, e_6, e_5, e_8, e_4) is a 5-edge configuration. Again by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1' b_2'\}$, (3) or the assertion of the lemma holds.

Now, since $\{v, a_8, a_2, b'_1, b'_2\}$ is not a cut in G^*, there exists $e = ab \in E(G)$ with $a \in V(A(a_8, a_2))$ and $b \in V(B - B[b'_1, b'_2])$. By (1), $b \in B[b_4, b'_1)$. Now $b = b_4$ by the choice of e_8. Hence, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. □

By (3), $\alpha(A, B) \leq 1$. Choose b''_2 so that $B[b''_2, b_7]$ is minimal. We may assume

(4) $b''_2 \notin B[b_7, b_2]$, and either $b_7 = b_2$ (in which case let $B_0 = B[b''_2, b_2]$) or $b_7 \neq b_2$ and $G_0 - (B[b_1, b''_2) \cup B[b_7, b_2))$ has a path B_0 from b''_2 to b_2.

Clearly, $b''_2 \notin B[b_7, b_2]$ as otherwise the conclusion of the lemma holds. Now suppose $b_7 \neq b_2$ and the desired path B_0 in $G_0 - (B[b_1, b''_2) \cup B[b_7, b_2))$ does not exist. Then there exist $b^*_2 \in V(B[b_7, b_2))$ and a separation (H_1, H_2) in G_0 such that $V(H_1 \cap H_2) = \{b_1, b^*_2, a_0\}$; so the conclusion of this lemma holds. □

(5) G has two nonadjacent edges from $B(b'_1, b_2]$ to $A[a_1, a_5]$.

For otherwise, $b'_1 = b_5$, and there exists $v \in \{a_7, b_7\}$ such that all edges in G from $B(b'_1, b_2]$ to $A[a_1, a_5]$ are incident with v. Then G has no edge from $B(b'_1, b_6]$ to $A(a_5, a_8)$, to avoid forming a double cross with e_4, e_5, e_8. Since $\{v, b'_1, b_2, a_8, a_2\}$ is not a cut in G^*, it follows form (1) that there exists $e = ab \in E(G)$ with $b \in V(B[b_4, b'_1))$ and $a \in V(A(a_8, a_2])$. By the choice of e_8, $b = b_4$. But then, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. □

Note that no two edges of G from $B(b'_1, b_2]$ to $A[a_1, a_4]$ can be parallel, as such edges would form a double cross with e_4, e_8. Therefore, by (5), G has two nonadjacent edges $e'_9 = a'_9 b'_9$, $e''_9 = a''_9 b''_9$ with $a'_9, a''_9 \in A[a_1, a_5]$ and $b'_9, b''_9 \in B(b'_1, b_2]$ such that b_1, b'_9, b''_9 occur on B in order, and a_1, a''_9, a'_9 occur on A in order. We further choose e'_9, e''_9 so that $A[a'_9, a_2] \cup B[b''_9, b_2]$ is minimal. Because of e_7, we have $a'_9 \in A[a_7, a_2]$ and $b''_9 \in B[b_7, b_2]$.

(6) G has two parallel edges $e' = a'b', e'' = a''b''$ with $b', b'' \in V(B(b_3, b'_1))$, $a', a'' \in V(A[a_4, a_2])$, and b_1, b', b'', b_2 on B in order.

We may assume $b_3 = b_4$; as otherwise e_4, e_8 give the desired edges for (6). Let $e = a_1 b \in E(G)$ with $b \notin \{b_1, b_2, b_3, b_7\}$. Then $b \notin B(b_1, b_3)$; otherwise, (e, e_4, e_5, e_6, e_7)

contradicts the choice of $\mathcal{P}$. Moreover, $b \notin B(b_3, b_7)$ to avoid the double cross e, e_4, e_7, e_8. So $b \in B(b_7, b_2)$.

Now, since (e, e_6, e_5, e_8, e_4) is a 5-edge configuration, $b_2'' \in B[b_6, b_7)$; or else, the desired separation of G_0 follows from Lemmas 2.1.9 and 2.5.3, the choice of $\{b_1', b_2'\}$, and the choice of b_2''.

Now, let $a^* \in A[a_1, a_2]$, such that G has an edge e^* from $b^* \in B(b_2'', b_7) \cup B(b_7, b_2)$ to a^*, subject to this, $A[a^*, a_2]$ is minimal, and subject to this, $B[b_2'', b^*]$ is minimal.

We claim that $a^* \notin A(a_5, a_2]$. For otherwise, suppose $a^* \in A(a_5, a_2]$. Now, if $b^* \in B(b_2'', b_7)$, then $(e_3, e_4, e_5, e^*, e_7)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. So $b^* \in B(b_7, b_2)$. If $a^* \in A(a_5, a_8)$, then e_4, e_5, e_8, e^* form a double cross; if $a^* \in A[a_8, a_2]$, then (e, e^*, e_5, e_8, e_4) is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We further claim that G has no edge from $A(a_1, a^*)$ to $B[b_1, b_3) \cup B(b_3, b_2'')$. (Recall that $b_3 = b_4$.) For otherwise, let $e' = a'b' \in E(G)$ with $a' \in A(a_1, a^*)$ and $b' \in B[b_1, b_3) \cup B(b_3, b_2'')$. Then $b' \notin B(b_3, b_2'')$ to avoid the double cross e_4, e_8, e', e^*. So $b' \in B[b_1, b_3)$. But then $a' \notin A(a_3, a^*)$ to avoid the double cross e_3, e_4, e', e_7. So $a' \in A[a_1, a_3]$, and (e', e_4, e_5, e_6, e_7) is a 5-edge configuration contradicting the choice of $\mathcal{P}$.

We may assume G has an edge e_7' from b_7 to $a_7' \in A(a^*, a_2]$ and an edge e_3' from b_3 to $a_3' \in A(a_1, a^*)$. For otherwise, G has a separation (H_1, H_2) of order 5, such that $V(H_1 \cap H_2) = \{a_1, a^*, v, b_2'', b_2\}$, $v \in \{b_3, b_7\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b_2'', b_2]) \subseteq V(H_2)$, a contradiction.

Then G has a separation (H_1, H_2) of order 6, such that $V(H_1 \cap H_2) = \{a_1, a^*, b_3, b_2'', b_7, b_2\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_1, a^*] \cup B[b_2'', b_2]) \subseteq V(H_2)$. Any two edges from $A[a_1, a^*]$ to $B[b_2'', b_2]$ are not parallel; or else, such two edges together with e_4, e_8 form a double cross. Moreover, by the choice of $\mathcal{P}$, we can further assume $a_7' \in A(a_5, a_2]$.

Now, assume $b^* \notin B(b_2'', b_7)$. Then since any two edges from $A[a_1, a^*]$ to $B[b_2'', b_2]$ are not parallel, then, combined with the choice of e^*, we have $(H_2, a_1, b_3, a^*, b_7, b_2'', b_2)$ is planar, a contradiction to Lemma 2.1.3.

So $b^* \in B(b_2'', b_7)$. But then $(e_7', e, e^*, e_6, e_3')$ is a 5-edge configuration. Now, by Lemmas 2.1.9 and 2.5.3 and by the choice of b_2'', G_0 has the desired separation. $\Box$

We choose e', e'' in (6) such that $B[b_3, b']$ is minimal and, subject to this, $B[b'', b_1']$ is minimal.

Suppose $G_0 - B(b_1, b_3] - B(b_1', b_2]$ has disjoint paths P_1, P_2 from b_1, a_0 to b', b'', respectively. Let $A' := P_2 \cup e'' \cup A[a'', a_2]$ and $B' := P_1 \cup e' \cup A[a_9', a'] \cup e_9' \cup B[b_9', b_2''] \cup B_0$. Now, since A, B is a good frame, the existence of A', B', $A[a_1, a_9''] \cup e_9'' \cup B[b_9'', b_2]$, and $A[a_1, a_3] \cup e_3 \cup B[b_1, b_3]$ shows $\alpha(A, B) = 2$, a contradiction.

Thus, such P_1, P_2 do not exist. Then G_0 has a separation (H_1, H_2) with $V(H_1 \cap H_2) = \{b_1^*, b_2^*\}$ such that $b_1^* \in B(b_1, b_3]$, $B[b_1^*, b''] \subseteq H_1$, and $\{a_0, b_1, b_2\} \subseteq H_2$. We may assume $b_2^* \in B[b'', b_1')$ as otherwise G_0 has the desired separation.

Since G^* is 6-connected, $\{b_1, b_1^*, b_2^*, b_1', a_0\}$ is not a cut in G; so there exists $e_0 = a_0 b_0 \in E(G)$ with $b_0 \in V(B(b_2^*, b_1'))$ and $a_0 \in V(A)$. By the choice of e', e'', $a_0 \in A[a_4, a'')$. So $(e_3, e'', e_0, e_6, e_7)$ is a 5-edge configuration. Now, by Lemma 2.1.9 and 2.5.3, and by the choice of $\{b_1', b_2'\}$ and the existence of $\{b_1^*, b_2^*\}$, G_0 has the desired separation.

Case 2. $a_8 \in A[a_1, a_3]$.

Note that if $b_3 = b_4$ we have symmetry between e_3 and e_4; so by Case 1, we may assume that if $b_3 = b_4$ then there exists $e_9 = a_4 b_9 \in E(G)$ with $b_9 \in B(b_4, b_1')$. Next, G has no edge from $B(b_3, b_7)$ to $A[a_1, a_3)$, to avoid the double cross e_3, e_9, e', e_7 (when $b_3 = b_4$) or e_3, e_4, e', e_7 (when $b_3 \neq b_4$). So $a_8 = a_3$, and all edges from $B(b_4, b_1')$ to A must end in $\{a_3, a_4\}$. Moreover, G has no edge from $B(b_4, b_7)$ to $A(a_4, a_2]$ to avoid forming a double cross with e_4, e_7, e_8. So $a_6 \notin A(a_4, a_2]$.

(1) For some $v \in \{a_4, b_4\}$, all edges from $B[b_1, b_1')$ to $A(a_3, a_2]$ are incident to v.

Now, we claim that G has no edge from $B[b_1, b_4)$ to $A(a_3, a_2]$. For, let $e = ab \in E(G)$ with $b \in B[b_1, b_4)$ and $a \in A(a_3, a_2]$. Then $a \in A[a_4, a_2]$, to avoid the double cross e, e_4, e_5, e_8. So $b = b_1$ by the choice of $\mathcal{P}$. Then $a \neq a_2$; so G has an edge $e_2 = a_2 b'$ with

$b' \in B(b_1, b_2)$. Then $b' \in B[b_7, b_2)$ to avoid the double cross e_2, e_7, e_8, e'. If $b_3 \neq b_4$ then (e_2, e_7, e_4, e_3, e) contradicts the choice of $\mathcal{P}$. So $b_3 = b_4$. Then e_9 is defined by (2.2.1). Hence, (e_2, e_7, e_9, e_3, e) contradicts the choice of $\mathcal{P}$.

Thus, suppose (1) fails, since all edges from $B(b_4, b_1')$ to A must end in $\{a_3, a_4\}$, then there exist $e' = a_4 b', e'' = a'' b_4$ with $a'' \in A(a_3, a_2] - a_4$ and $b' \in B(b_4, b_1')$. By the choice of $\mathcal{P}$, $a'' \in A(a_3, a_4)$. So e_8, e', e'', e_7 form a double cross, a contradiction. □

(2) $a_1 = a_3$.

For, suppose $a_1 \neq a_3$. Then there exists $e_1 = a_1 b \in E(G)$ with $b \in V(B(b_1, b_2))$. Note that $b \notin B(b_3, b_7)$ by observation above (1), and $b \notin B(b_1, b_4]$ as otherwise $(e_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. So $b \in B[b_7, b_2)$. Moreover, $b_3 = b_4$, for, otherwise, $(e_7, e_1, e_8, e_4, e_3)$ contradicts the choice of $\mathcal{P}$. Thus the edge e_9 is defined, and hence $v = a_4$.

Now G has no edge from $B[b_1', b_7)$ to $A(a_1, a_7)$. For such an edge and e_1, e_7, e_9, e_3 form a 5-edge configuration. Hence, by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

Thus, $a_6 \in A(a_7, a_4]$ by (1). So $(e_6, e_1, e_5, e_9, e_3)$ is a 5-edge configuration. If $b_2' \neq b_2$ then by Lemmas 2.1.9 and 2.5.3 and by the choice of $\{b_1', b_2'\}$, G_0 has the desired separation.

So $b_2' = b_2$. Since G^* is 6-connected, $\{b_1, b_2, b_1', a_3, a_4\}$ is not a cut in G. Hence, there exists $e^* = a^* b^* \in E(G)$ with $a^* \in V(A[a_1, a_2]) \setminus \{a_3, a_4\}$ and $b^* \in V(B(b_1, b_1'))$. By (1) and by the existence of e_9, $a^* \in A[a_1, a_3)$. Then $b^* \notin B(b_1, b_3]$; otherwise, $(e^*, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then, $b^* \in B(b_3, b_1')$, and e^*, e_3, e_6, e_7 form a double cross. □

Let $e_2 = a_2' b' \in E(G)$ with $a_2' \in V(A)$ and $b' \in V(B(b_1', b_2'))$, such that $A[a_2, a_2']$ is minimal. Since G^* is 6-connected, $\{b_1', b_2', a_1, a_2'\}$ is not a cut in G; so there exists $e_0 = a_0 b_0 \in E(G)$ with $a_0 \in V(A(a_1, a_2'))$ and $b_0 \in V(B - B[b_1', b_2'])$.

We claim that $b_0 \in B[b_1, b_1')$ for every choice of e_0. For, otherwise, $b_0 \in B(b_2', b_2]$.

Then $a_0 \in A(a_1, a_4)$ to avoid the double cross e_4, e_8, e_2, e_0. Also, $a_6 \in A[a_5, a_0]$; otherwise $(e_3, e_4, e_5, e_6, e_0)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_7 \in A[a_6, a_0]$; or else $(e_3, e_4, e_6, e_7, e_0)$ contradicts the choice of $\mathcal{P}$. But this shows that $a_6 \in A[a_5, a_7]$, a contradiction.

Therefore, by (1), $\{a_1, a_2', b_1', b_2', v\}$ is a cut in G^* separating $\{a_0, a_1, a_2, b_1, b_2\}$ from $A[a_1, a_2'] \cup B[b_1', b_2']$, a contradiction. $\square$

Thus by Lemmas 2.5.3, 2.5.5, and 2.5.6, G_0 has a separation (G_1, G_2) with $|V(G_1) \cap V(G_2)| \leq 3$, $|V(G_1 - G_2)| \geq 1$, $\{a_0, a_1, a_2\} \subseteq V(G_1)$, and $B[b_1', b_2'] \subseteq G_2$ where $b_1', b_2' \in V(G_1) \cap V(G_2)$, such that one of the following holds:

(a) $|V(G_1) \cap V(G_2)| = 3$, $b_1' \in B[b_1, b_4]$, $b_2' \in B[b_7, b_2]$, and G_0 has a path from a_0 to $B(b_1', b_2')$ and internally disjoint from B. In this case, let $t_1 := b_1'$, $t_2 := b_2'$, and $a_0' = t_0 \in V(G_1 \cap G_2) \setminus \{b_1', b_2'\}$.

(b) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_1, b_4]$, and $b_2' \in B[b_7, b_2]$. In this case, let $t_0 = t_1 := b_1'$, and $t_2 := b_2'$.

(c) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B[b_1, b_4]$, $b_2' \in B[b_6, b_7)$, and G has no edge from $B(b_2', b_7)$ to $A - a_7$. In this case, let $t_1 := b_1'$ and $t_0 = b_2'$. Moreover, if G has no edge from $B(b_2', b_7)$ to a_7 then let $t_2 := b_7$, and if G has an edge f_7 from $b_7^* \in B(b_2', b_7)$ to a_7 then let $t_2 := a_7$, $B(t_1, t_2) := B(b_1', b_2']$ and $B(t_2, b_2] := B[b_7, b_2]$.

(d) $|V(G_1) \cap V(G_2)| = 2$, $b_1' \in B(b_4, b_5]$, $b_2' \in B[b_7, b_2]$, and G has no edge from $B(b_4, b_1')$ to $A - a_4$. In this case, let $t_0 := b_1'$ and $t_2 := b_2'$. Moreover, if G has no edge from $B(b_4, b_1')$ to a_4 then let $t_1 := b_4$, and if G has an edge f_4 from $b_4^* \in B(b_4, b_1')$ to a_4 then let $t_1 := a_4$, $B(t_1, t_2) := B[b_1', b_2')$ and $B[b_1, t_1) := B[b_1, b_4]$.

We choose b_1', b_2' so that b_1', b_2' satisfy (a) or (b) whenever possible, subject to this, $B[b_1, b_1']$ is minimal, and subject to this, $B[b_7, b_2']$ is minimal.

Let $f_i = a_i^* b_i^* \in E(G)$, $i \in [2]$, with $a_i^* \in V(A)$ and $b_i^* \in V(B(t_1, t_2))$ such that $A[a_1^*, a_2^*]$ is maximal. Then $A[a_5, a_6] \subseteq A[a_1^*, a_2^*]$. Without loss of generality, we may assume that a_1, a_1^*, a_2^*, a_2 occur on A in order.

Lemma 2.5.7 *$G - e_4$ has an edge from $B[b_1, t_1)$ to $A(a_1^*, a_2^*)$.*

Proof. For, suppose $G - e_4$ has no edge from $B[b_1, t_1)$ to $A(a_1^*, a_2^*)$. Then, since $\{t_0, t_1, t_2, a_1^*, a_2^*\}$ is not a cut in G^* separating $A(a_1^*, a_2^*) \cup B(t_1, t_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, there exists $e_8 = a_8 b_8 \in E(G)$ with $b_8 \in V(B(t_2, b_2])$ and $a_8 \in V(A(a_1^*, a_2^*) - t_2)$. Obviously, $b_8 \in B(b_2', b_2] \cap B[b_7, b_2]$.

We claim that $a_8 \in A(a_3, a_4)$. For, otherwise, $a_8 \in A(a_1, a_3] \cup A[a_4, a_2)$. If $a_8 \in A(a_1, a_3]$, then $a_1^* \in A[a_1, a_3)$, and so e_3, f_1, e_5, e_8 force a double cross, or $(f_1, e_4, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Therefore, $a_8 \in A[a_4, a_2)$. Then $b_2^* \in B(b_1', b_4]$; otherwise e_4, e_5, f_2, e_8 force a double cross. But now, $(e_3, f_2, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

If $b_8 \in B(b_7, b_2]$ then $(e_3, e_4, e_5, e_6, e_8)$ (when $a_6 \notin A[a_5, a_8]$) or $(e_3, e_4, e_6, e_7, e_8)$ (when $a_6 \in A[a_5, a_8]$) contradicts the choice of $\mathcal{P}$.

Hence $b_8 = b_7$ and, thus, $t_2 = a_7 \neq a_8$ and G has an edge $f_7 = a_7 b_7^*$ with $b_7^* \in V(B(b_2', b_7))$. Let $e = a_8 b \in E(G)$ with $b \in V(B[b_1, b_2]) \setminus \{b_4, b_7\}$, which exists as G^* is 6-connected.

We claim that $b \in B[b_1, b_4)$. Note that $b \notin B(b_2', b_7)$ (as $t_2 = a_7$) and $b \notin B(b_7, b_2]$ (as $b_8 = b_7$). So if the claim fails then $b \in B(b_4, b_2']$; now (e_3, e_4, e, f_7, e_8) contradicts the choice of $\mathcal{P}$.

Thus, $a_8 \in A(a_3, a_7)$ to avoid the double cross e, e_4, f_7, e_8. Then $a_7 \in A[a_1, a_5]$; otherwise, $(e_3, e_4, e_5, f_7, e_8)$ contradicts the choice of $\mathcal{P}$. Now $a_6 \in A(a_5, a_2]$, for, if $a_6 \in A[a_1, a_8)$ then e_4, e_6, e_8, e form a double cross, and if $a_6 \in A[a_8, a_7)$ then $(e_3, e_4, e_6, f_7, e_8)$ contradicts the choice of $\mathcal{P}$.

Suppose there exists $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_5))$ and $b_9 \in V(B(b_4, b_5])$. Then $a_9 \notin A[a_1, a_8)$ to avoid the double cross e, e_4, e_8, e_9. Moreover, $a_9 \notin A[a_8, a_7)$, or else

$(e_3, e_4, e_9, f_7, e_8)$ contradicts the choice of $\mathcal{P}$. So $a_9 \in A[a_7, a_5)$. Now $(e_3, e_4, e_9, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Hence, G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$. By Lemma 2.5.4 and by the existence of $\{b'_1, b'_2\}$, there exists $e_9 = a_9 b_9 \in E(G)$ with $a_9 \in V(A(a_5, a_2])$ and $b_9 \in V(B[b_7, b_2])$. Then $(e_9, e_8, f_7, e_6, e_5)$ is a 5-edge configuration. Then G_0 has a cut $\{b''_1, b''_2\}$ or $\{b''_1, b''_2, a''_0\}$ satisfying the conclusion of Lemma 2.5.3 (with respect to $(e_9, e_8, f_7, e_6, e_5)$), such that b_1, b''_1, b''_2, b_2 occur on B in order. But then, by Lemma 2.1.9, G_0 has a cut that would contradict the choice of $\{b'_1, b'_2\}$. □

Thus, by Lemma 2.5.7, there exists $e_8 = a_8 b_8 \in E(G - e_4)$ with $b_8 \in V(B[b_1, t_1))$ and $a_8 \in V(A(a^*_1, a^*_2))$. Note that $b_8 \in B[b_1, b_4] \cap B[b_1, b'_1)$.

Lemma 2.5.8 $a_8 \in A(a^*_1, a_5]$.

Proof. For otherwise, $a_8 \in A(a_5, a^*_2)$, and we choose e_8 so that $A[a_8, a_2]$ is maximal. Then

(1) $b_8 \notin B(b_1, b_4]$ for all choices of b_8.

First, suppose $b_8 \in B(b_1, b_4)$. Then $a_8 \notin A(a_5, a_7]$ to avoid the double cross e_8, e_4, e_5, e_7. Now, $b_3 = b_4$ and $a_8 \in A[a_1, a_4)$; otherwise, $(e_3, e_8, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then, e_3, e_4, e_7, e_8 form a double cross.

Now assume $b_8 = b_4$. Then $t_1 = a_4$ and there exists $f_4 = a_4 b^*_4 \in E(G)$ with $b^*_4 \in V(B(b_4, b'_1))$. Note that $a_8 \in A(a_5, a_4)$; otherwise, by $e_8 \neq e_4$, we have $a_8 \in A(a_4, a_2]$ and $(e_3, e_8, e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

G has no edge from $A(a_5, a_4)$ to $B(b_5, b_2]$, to avoid forming a double cross with e_5, e_8, f_4. Hence, $a_7 \in A(a_3, a_5]$ and $a_6 \notin A(a_5, a_4)$. Moreover, $a_6 \notin A[a_1, a_7)$ to avoid the double cross e_6, e_7, e_8, f_4. So $a_6 \in A[a_4, a_2]$.

Since $d_G(a_8) \geq 6$, there exists $e'_8 = a_8 b'_8 \in E(G)$ with $b'_8 \in V(B[b_1, b_2]) - \{b_1, b_4, b_5\}$. Since $b_8 \notin B(b_1, b_4)$ and G has no edge from $A(a_5, a_4)$ to $B(b_5, b_2]$, then $b'_8 \in B(b_4, b_5)$. But then, $(e_3, e_4, e'_8, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. □

Hence, $b_8 = b_1$ and $b_1' \neq b_1$. Now, $a_8 \in A[a_4, a_2^*)$ to avoid the double cross e_8, e_4, e_3, e_7. And $b_2^* \in B[b_7, b_2')$ to avoid the double cross e_8, f_2, e_3, e_7. Then $b_3 = b_4$; otherwise, $(f_2, e_7, e_4, e_3, e_8)$ contradicts the choice of $\mathcal{P}$.

Note that $a_5 \in A[a_1, a_7]$, or else $(f_2, e_7, e_5, e_3, e_8)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_6 \in A[a_1, a_5)$, as, otherwise, e_8, e_6, e_3, e_7 (when $a_6 \in A(a_8, a_2]$) would form a double cross, or $(f_2, e_7, e_6, e_3, e_8)$ (when $a_6 \in A[a_5, a_8]$) contradicts the choice of $\mathcal{P}$.

(2) G has no cross from $B[b_6, b_2]$ to $A[a_5, a_2]$ and G has no edge from $B(b_6, b_2]$ to $A[a_1, a_5)$.

Note that G has no cross from $B[b_6, b_2]$ to $A[a_5, a_2]$, to avoid forming a double cross with e_5, e_6. Now suppose there exists $e = ab \in E(G)$ with $b \in V(B(b_6, b_2])$ and $a \in V(A[a_1, a_5))$. Then $b = b_2$; or else, (e_3, e_4, e_5, e, e_7) (when $b \notin B(b_6, b_7)$) or (f_2, e, e_5, e_3, e_8) (when $b \in B[b_7, b_2)$) contradicts the choice of $\mathcal{P}$. But then $a \neq a_1$, and e, e_8 and two edges from a_1, a_2 to $B(b_1, b_2)$ would form a double cross. $\square$

(3) G has no edge from $B(b_1, b_3)$ to A.

For, otherwise, let $e = ab \in E(G)$ with $a \in A$ and $b \in B(b_1, b_3)$. Then $a \in A[a_4, a_8]$; or else, (f_2, e_7, e_4, e, e_8) contradicts the choice of $\mathcal{P}$. But now, (e, e_3, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. $\square$

(4) G has no edge from $A(a_4, a_2]$ to $B(b_1, b_7)$.

For, otherwise, let $e = ab \in E(G)$ with $a \in A(a_4, a_2]$ and $b \in B(b_1, b_7)$. Then $b \notin B(b_4, b_7)$ to avoid the double cross e_4, e_6, e_7, e. But then $b \in B(b_1, b_4]$, and (e, e_3, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. $\square$

Let $e^* = a_2 b^* \in E(G)$, such that $b^* \in B(b_1, b_2)$, and $B[b^*, b_2]$ is minimal. Then by (2) and (4), $b^* \in B[b_7, b_2)$ and G has no edge from $B(b^*, b_2]$ to A.

Let $e' = a'b' \in E(G)$ with $a' \in A(a_8, a_2]$ and $b' \in B(b_6, b_2]$, such that $B[b', b_2]$ is maximal. Note that e' exists because of e^*. And $b' \in B[b_7, b^*]$ by (2).

Now, by (2), (4), and the choice of e^*, e', we have

(5) G has no edge from $B(b^*, b_2]$ to A and no edge from $B(b_1, b')$ to $A(a_8, a_2]$.

(6) G has no edge from b_1 to $A[a_1, a_8)$.

For, suppose there exists $e = ab_1 \in E(G)$ with $a \in V(A[a_1, a_8))$. Then, by the choice of e_8, $a \notin A(a_5, a_8)$. Hence, $a \in A[a_1, a_5]$. Since $a \neq a_1$, there exists $e_0 = a_1b_0 \in E(G)$ with $b_0 \in V(B(b_1, b_2))$. Then $b_0 \in B[b_7, b_2)$ to avoid the double cross e_0, e_4, e_7, e. So (e_0, e^*, e_5, e_4, e) contradicts the choice of $\mathcal{P}$. $\square$

(7) If there exists $f_8' = a_8'b_8' \in E(G)$ with $a_8' \in V(A[a_5, a_2])$ and $b_8' \in V(B(b_6, b_2])$, then G has no edge from $B(b_4, b_8')$ to $A(a_8', a_2]$.

For, suppose such f_8' exists, and let $f_9' = a_9'b_9' \in E(G)$ with $a_9' \in A(a_8', a_2]$ and $b_9' \in B(b_4, b_8')$. Then $b_9' \notin B(b_5, b_8')$ to avoid the double cross e_5, e_6, f_8', f_9'. So $b_9' \in B(b_4, b_5]$. Moreover, $b_9' \notin B(b_4, b_5)$; otherwise, $(e_3, e_4, f_9', e_6, e_7)$ contradicts the choice of $\mathcal{P}$. So $b_9' = b_5$. Now, we see that $a_7 \in A[a_5, a_9')$; or else, $(e_3, e_4, f_9', e_6, e_7)$ contradicts the choice of $\mathcal{P}$. But then $(e^*, e_7, f_9', e_3, e_8)$ is a 5-edge configuration contradicting the choice of $\mathcal{P}$. $\square$

(8) There do not exist $b'' \in V(B[b_6, b'])$ and a cut S of G_0 such that $|S| \leq 3$, $\{b_3, b''\} \subseteq S$, and S separates $B[b_3, b'']$ from $\{a_0, b_1, b_2\}$.

For, suppose such b'' and S do exist. Let $f_9' = a_9'b_9' \in E(G)$, such that $a_9' \in V(A[a_1, a_2])$, $b_9' \in V(B(b_3, b''))$, and subject to this, $A[a_9', a_2]$ is minimal. Then $a_9' \in A[a_5, a_2]$, by the existence of e_5.

We claim that $a_9' \notin A(a_8, a_2]$, and so by (6), G has no edge from b_1 to $A[a_1, a_9')$. For otherwise, $b_9' \notin B(b_3, b_7)$ to avoid the double cross e_6, e_7, e_8, f_9'. But then $b_9' \in B[b_7, b')$, and f_9' contradicts the choice of e'.

By (2) and (7), G has no edge from $B(b'', b_2]$ to $A[a_1, a_9')$. Thus, $S \cup \{a_1, a_9'\}$ is a cut in G^* separating $A[a_1, a_9'] \cup B[b_3, b'']$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction. $\square$

Since (e', e_6, e_5, e_3, e_8) is a 5-edge configuration, G_0 has a cut $S' := \{b_1'', b_2''\}$ or $S' := \{b_1'', b_2'', a_0''\}$ satisfying the conclusion of Lemma 2.5.3 (with respect to (e', e_6, e_5, e_3, e_8)), such that b_1, b_1'', b_2'', b_2 occur on B in order.

Case 1. Conclusions (i), or (ii), or (iii) of Lemma 2.5.3 holds for S' and (e', e_6, e_5, e_3, e_8).

Since $(e_3, e_4, e_5, e_6, e_7)$ is a 5-edge configuration, G_0 has a cut $S^{\#} := \{b_1^{\#}, b_2^{\#}\}$ or $S^{\#} := \{b_1^{\#}, b_2^{\#}, a_0^{\#}\}$ satisfying the conclusion of Lemma 2.5.3 (with respect to $(e_3, e_4, e_5, e_6, e_7)$), such that $b_1, b_1^{\#}, b_2^{\#}, b_2$ occur on B in order.

We may assume conclusion (iv) of Lemma 2.5.3 holds for $S^{\#}$ and $(e_3, e_4, e_5, e_6, e_7)$, and so $b_1^{\#} \in B(b_4, b_5]$ and $b_2^{\#} \in B[b_7, b_2]$. For otherwise, assume conclusions (i), or (ii), or (iii) of Lemma 2.5.3 holds for $S^{\#}$ and $(e_3, e_4, e_5, e_6, e_7)$. Then by the choice of $\{b_1', b_2'\}$ and $b_1' \neq b_1$, and by Lemma 2.1.8 and 2.1.9, we could find a cut $\{b_3, b''\}$ or $\{b_3, b'', a''\}$ with $b'' \in B[b_6, b']$ in G_0, which separates $B[b_3, b'']$ from $\{a_0, b_1, b_2\}$, a contradiction to (8).

Suppose conclusion (i) of Lemma 2.5.3 holds for $\{b_1'', b_2'', a_0''\}$ and (e', e_6, e_5, e_3, e_8). Then $b_2'' \in B[b_6, b_7)$ by $b_1' \neq b_1$ and the choice of $\{b_1', b_2'\}$. Moreover, by Lemma 2.1.9, $b_2^{\#} = b_2$, and $b_1^{\#}, b_2'', b_2, a_0$ are incident with a finite face of G_0. Let $f_8' = a_8'b_8' \in E(G)$ with $a_8' \in V(A[a_1, a_2])$ and $b_8' \in V(B(b_2'', b_2])$, such that $A[a_8', a_2]$ is maximal. Now, by (2), (3), and (7), G has a separation (H_1, H_2), such that $V(H_1 \cap H_2) = \{b_1, b_2, b_4, b_2'', a_8'\}$, $\{a_0, a_1, b_1, b_2\} \subseteq V(H_1)$, and $V(A[a_8', a_2] \cup B[b_2'', b_2]) \subseteq V(H_2)$, a contradiction.

Now suppose conclusion (ii) of Lemma 2.5.3 holds for $\{b_1'', b_2''\}$ and (e', e_6, e_5, e_3, e_8). So $b_1'' = b_1$ and $b_2'' \in B[b_6, b']$. Then by Lemma 2.1.9, $\{b_1, b_2^{\#}\}$ is a cut in G_0 separating $B[b_1, b_2^{\#}]$ from $\{b_1, b_2, a_0\}$, which contradicts the choice of $\{b_1', b_2'\}$ (as $b_1' \neq b_1$).

So conclusion (iii) of Lemma 2.5.3 holds for $\{b_1'', b_2''\}$ and (e', e_6, e_5, e_3, e_8). Now $b_1'' \in B(b_1, b_3]$ and $b_2'' \in B[b_6, b']$. Then by Lemma 2.1.9, $\{b_1'', b_2^{\#}\}$ is a cut in G_0 separating $B[b_1'', b_2^{\#}]$ from $\{b_1, b_2, a_0\}$. Let $f_9' = a_9'b_9' \in E(G)$, with $a_9' \in V(A[a_4, a_2])$ and $b_9' \in V(B[b_4, b_2^{\#}))$, such that $A[a_9', a_2]$ is minimal. If G has no edge from $B(b_2^{\#}, b_2]$ to $A[a_1, a_9')$ then $\{b_1, b_1'', b_2^{\#}, a_9'\}$ is a cut in G^* separating $\{a_0, a_2, b_1, b_2\}$ from $A[a_1, a_9'] \cup B[b_1'', b_2^{\#}]$, a contradiction. So there exists $f_8' = a_8'b_8' \in E(G)$ with $a_8' \in V(A[a_1, a_9']))$ and $b_8' \in V(B(b_2^{\#}, b_2])$. Then $a_8' \notin A[a_5, a_4)$; or else, $(e_3, e_4, e_5, e_6, e_8')$ contradicts the choice of $\mathcal{P}$. So $a_8' \in A[a_4, a_2]$ by (2), and $b_9' = b_4$ by (7). But now, $(e_3, f_9', e_5, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Case 2. Conclusion (iv) of Lemma 2.5.3 holds for S' and (e', e_6, e_5, e_3, e_8).

Then $b_2'' \in B[b_5, b_6)$, $b_1'' = b_1$, and $\{b_1, b_2''\}$ is a cut in G_0 separating $B[b_1, b_2'']$ from $\{a_0, b_1, b_2\}$. By Lemma 2.1.9, the choice of $\{b_1', b_2'\}$, and $b_1' \neq b_1$, we have $b_2' = b_2$, $b_1' \in B(b_1, b_3]$, and b_1', b_2'' are cut vertices of G_0 separating b_1 from $\{a_0, b_2\}$. So $\alpha(A, B) \leq 1$.

Recall $e^* = a_2b^*$ with $B[b^*, b_2]$ minimal. If $b^* = b_7$, then, by (4) and (5), $\{b_1, b_7, a_4\}$ is a cut in G^* separating $\{a_0, a_1, b_1, b_2\}$ from $A(a_4, a_2]$, a contradiction. So $b^* \neq b_7$. Then $b^* \in B(b_7, b_2]$. Note that no finite face of G_0 is incident with both b_2'' and some vertex $u \in B[b^*, b_2)$; or else, $\{b_1'', b_2'', u\}$ is a 3-cut in G_0 separating $B[b_1'', u]$ from $\{a_0, b_1, b_2\}$, contradicting the choice of $\{b_1', b_2'\}$.

We claim that $G_0 - B[b_1, b_2''] - B[b^*, b_2)$ has disjoint paths B_2, A_0 from b_2, a_0 to b_7, b_6, respectively. For otherwise, since we may assume that Case 1 does not hold, it follows from the planar structure of G_0 and the choice of $\{b_1', b_2'\}$ that there exist $u_0 \in V(G_0)$, $u_2 \in B[b^*, b_2)$, such that $\{b_2'', u_0, u_2\}$ is a cut in G_0 separating $B[b_1'', b_2'') \cup B(b_2'', u_2)$ from $\{a_0, b_2\}$. By (5), $\{b_2'', u_0', u_2\}$ is a cut in G^* separating $\{a_0, b_2\}$ from $\{a_1, a_2, b_1\}$, a contradiction.

Now, let $A' := A[a_1, a_6] \cup e_6 \cup A_0$ and $B' := B[b_1, b_5] \cup e_5 \cup A[a_5, a_7] \cup e_7 \cup B_2$. Then the existence of A', B', $e_8 \cup A[a_8, a_2]$, and $e^* \cup B[b', b_2]$ implies $\alpha(A, B) = 2$ (by Lemma 2.2.1), a contradiction. $\square$

Thus by Lemma 2.5.8, $a_8 \in A(a_1^*, a_5]$ for all choices of e_8. Choose e_8 so that $A[a_8, a_5]$ is minimal and, subject to this, $B[b_8, b_1']$ is minimal. Then G has no edge from $B[b_1, b_4] \cap B[b_1, b_1')$ to $A(a_8, a_2^*)$.

(1) G has no cross from $B[b_1, b_4]$ to $A[a_1, a_5]$; so $b_8 \in B[b_3, b_4]$.

For, such a cross would form a double cross with e_4, e_5. $\square$

(2) G has no edge from $B(b_8, b_7)$ to $A[a_1, a_8) \cap A[a_1, a_7)$; so $b_1^* \in B[b_7, b_2)$ if $a_8 \in A[a_1, a_7]$.

For, such an edge would form a double cross with e_4, e_7, e_8 (when $b_8 \neq b_4$) or f_4, e_7, e_8 (when $t_1 = a_4$ and $b_8 = b_4$). $\square$

(3) $a_7 \in A[a_1, a_5]$.

For, suppose $a_7 \in A(a_5, a_2]$. Then $b_1^* \in B[b_7, b_2)$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$). Now, we may assume $t_1 = a_4$ and $b_8 = b_4$; otherwise, $b_8 \in B[b_1, b_4)$ and $(f_1, e_7, e_5, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then $(f_1, e_7, e_5, f_4, e_8)$ is a 5-edge configuration. So by Lemmas 2.1.9 and 2.5.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\square$

(4) G has no edge from $B(b_5, b_7)$ to $A[a_1, a_7)$, and so $a_6 \in A(a_5, a_2]$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_7))$ and $b_9 \in V(B(b_5, b_7))$. Then $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2)$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$) and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration. Now $t_1 = a_4$ and $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. So by Lemmas 2.1.9 and 2.5.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\square$

(5) G has no edge from $B(b_4, b_5]$ to $A[a_1, a_5)$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A[a_1, a_5))$ and $b_9 \in V(B(b_4, b_5])$. Then $a_9 \notin A[a_7, a_5)$; otherwise, $(e_3, e_4, f_9, e_6, e_7)$ contradicts the choice of $\mathcal{P}$. Moreover, $a_8 \in A[a_1, a_9]$ and $b_1^* \in B[b_7, b_2)$ by (2). So $b_7 \neq b_2$ (as $b_1^* \neq b_2$) and $(f_1, e_7, e_9, f_4, e_8)$ is a 5-edge configuration. Now, $t_1 = a_4$ and $b_8 = b_4$; otherwise, $(f_1, e_7, e_9, e_4, e_8)$ contradicts the choice of $\mathcal{P}$. But then, $b_9 = b_5$ and by Lemmas 2.1.9 and 2.5.3, G_0 has a cut contradicting the choice of $\{b_1', b_2'\}$. $\square$

(6) G has no edge from $B(b_6, b_2]$ to $A(a_5, a_2]$.

For, otherwise, let $e_9 = a_9b_9 \in E(G)$ with $a_9 \in V(A(a_5, a_2])$ and $b_9 \in V(B(b_6, b_2])$. Then $b_9 \in B[b_7, b_2]$; or else, $(e_3, e_4, e_5, e_9, e_7)$ contradicts the choice of $\mathcal{P}$.

Suppose $b_9 = b_2$. Then $a_9 \neq a_2$ and let $e = a_2b \in E(G)$ with $b \in V(B(b_1, b_2))$ and $b \neq b_4$. If $b \in B(b_1, b_4)$ then (e_3, e, e_5, f_1, e_9) contradicts the choice of $\mathcal{P}$; if $b \in B(b_4, b_2)$ then e_8, e_9, f_1, e form a double cross, a contradiction.

So $b_9 \in B[b_7, b_2)$ and $b_7 \neq b_2$. So $(e_9, e_7, e_5, e_4, e_8)$ (when $a_7 \in A[a_1, a_8)$) or $(e_9, f_1, e_5, e_4, e_8)$ (when $a_8 \in A[a_1, a_7]$ and by (2)) is a 5-edge configuration. Hence, by the choice of $\mathcal{P}$, $t_1 = a_4$ and $b_8 = b_4$. Now by Lemma 2.1.9 and 2.5.3, G_0 has a cut contradicting the choice of $\{b'_1, b'_2\}$. $\square$

Now, by (3)–(6) and by Lemma 2.5.4,

(7) G_0 does not contain a cut $\{b''_1, b''_2\}$ separating $B[b''_1, b''_2]$ from $\{a_0, b_1, b_2\}$ with $b''_1 \in B[b_1, b_4]$ and $b''_2 \in B[b_6, b_2]$.

By (7), we have

(8) (b) and (c) do not hold.

(9) G has no edge from $B[b_1, b_4)$ to $A(a_5, a_2]$.

For, suppose there exists $e = ab \in E(G)$ with $b \in V(B[b_1, b_4))$ and $a \in V(A(a_5, a_2])$. If $b \in B(b_1, b_4)$ then $a \in A(a_5, a_4)$ and $b_3 \in B(b, b_4]$; or else, (e_3, e, e_5, e_6, e_7) contradicts the choice of $\mathcal{P}$. But then e_3, e_4, e, e_5 form a double cross.

So $b = b_1$ and, hence, $a \neq a_2$. Let $e_0 = a_2b_0 \in E(G)$ with $b_0 \in V(B(b_1, b_2))$. By (6), $b_0 \in B(b_1, b_7)$. But then e_0, e, e_3, e_7 form a double cross, a contradiction. $\square$

(10) G has no parallel edges from $A[a_1, a_8]$ to $B[b_4, b_2]$ and no parallel edges from $A[a_1, a_5]$ to $B[b_6, b_2]$.

For, such parallel edges would form a double cross with e_4, e_8 or e_5, e_6. $\square$

Let $e'_7 = a'_7b'_7 \in E(G)$ with $a'_7 \in A[a_1, a_7]$ and $b'_7 \in B[b_7, b_2]$, such that $A[a_1, a'_7] \cup B[b'_7, b_2]$ is minimal. Then

(11) $a_7' \in A[a_1, a_8)$, and G has no edge from $B(b_7', b_2]$ to A.

For, if $a_7' \notin A[a_1, a_8)$ then, since $a_1^* \in A[a_1, a_8)$, $b_1^* \in B(b_8, b_7')$ by the choice of e_7'; so e_8, e_4, f_1, e_7' form a double cross, a contradiction. Thus, by (6) and (10) and by the choice of e_7', G has no edge from $B(b_7', b_2]$ to A. $\square$

Let $e' = a'b' \in E(G)$ with $a' \in A[a_1, a_5]$ and $b' \in B[b_1, t_1)$, such that $A[a_1, a'] \cup B[b_1, b']$ is minimal. By (1) and (9) and by the choice of e', we have

(12) e', e_8 do not form a cross, and G has no edge from $B[b_1, b')$ to A, and no edge from $B(b', b_8)$ to $A[a_1, a') \cup A(a_8, a_2]$.

(13) If (d) holds then there does not exist a 3-cut $\{b_1'', b_2'', a_0''\}$ in G_0 with $b_1'' \in B[b_1, b_4]$ and $b_2'' \in B(b_5, b_2)$, which separates $B[b_1'', b_2'']$ from $\{a_0, b_1, b_2\}$.

For, suppose (d) holds and the cut $\{b_1', b_2'', a_0''\}$ in (13) exists. Then $b_1' \in B(b_4, b_5]$, $b_2' \in B[b_7, b_2]$, and G has no edge from $B(b_4, b_1')$ to $A - a_4$. Now, by the choice of $\{b_1', b_2'\}$ and by Lemma 2.1.9, $b_1'' = b_1, b_2'' \in B(b_5, b_7)$, $a_0'' = a_0$, $b_2' = b_2$, and $\alpha(A, B) \leq 1$.

By the choice of $\{b_1', b_2'\}$ and by the planar structure of G_0, $G_0 - a_0 - B[b_7', b_2)$ contains a path B_2 from b_2 to b_2''. Let $e_4' = a_4 b_4' \in E(G)$ with $b_4' \in B[b_4, b_1')$ such that $B[b_4', b_1']$ is minimal. Since $b_8 \in B[b_1, t_1)$, then $b_8 \neq b_4'$.

We claim that if $b_4' \neq b_4$ then G has no edge from $B[b_1, b_4')$ to $A(a_5, a_2] - a_4$. For, suppose $b_4' \in B(b_4, b_1')$ and there exists $e = ab \in E(G)$ from $b \in V(B[b_1, b_4'))$ to $a \in V(A(a_5, a_2] - a_4)$. Now $b = b_4$ by (9) and (d). So $a \in A(a_5, a_4)$ by the choice of $\mathcal{P}$. Let $e_0 = ab_0 \in E(G)$ with $b_0 \in V(B[b_1, b_2]) \setminus \{b_4, b_5\}$. Then $b_0 \notin B[b_1, b_4)$ by (9). Moreover, $b_0 \notin B(b_5, b_2]$ to avoid the double cross e, e_0, e_4', e_5. So $b_0 \in B(b_4, b_5)$. If $a_6 \in A(a_5, a_4)$ then e, e_6, e_4', e_5 form a double cross; if $a_6 \in A[a_4, a_2]$ then $(e_3, e_4, e_0, e_6, e_7)$ contradicts the choice of $\mathcal{P}$.

Hence, by the choice of e_8, (1), (9), and (d), if $b_4' = b_4$, then G has no edge from $B(b_8, b_4')$ to A; if $b_4' \neq b_4$, then G has no edge from $B(b_8, b_4')$ to $A - a_4$.

Now e' is not adjacent with e_8. For, suppose v is a vertex incident with both e' and e_8. Then, by (12), (d), and our previous analysis, $\{b_1, v, b_4, b_1', b_2\}$ (when $b_4' = b_4$) or $\{b_1, v, a_4, b_1', b_2\}$ (when $b_4' \neq b_4$) is a cut in G^* separating a_0 from A, a contradiction.

$G_0 - B(b_1, b'] - B[b_1', b_2]$ contains disjoint paths B_1, A_0 from b_1, a_0 to b_8, b_4', respectively. For, suppose there exists a cut vertex v in $G_0 - B(b_1, b'] - B[b_1', b_2]$ separating $\{b_1, a_0\}$ from $\{b_8, b_4'\}$. Then $v \notin B[b', b_8]$; otherwise, v and b_1' are incident with some finite face of G_0, and so $\{v, b_1', b_2'\}$ is a 3-cut in G_0 separating $B[v, b_2']$ from $\{a_0, b_1, b_2\}$, contradicting the choice of $\{b_1', b_2'\}$. Moreover, $v \notin B[b_4', b_1']$; for otherwise, there exists $v_1 \in V(B(b_1, b'])$ such that v_1, v are incident with some finite face of G_0 and, by (12), (d), and the choice of e_4', $\{v_1, v, b_1'\}$ is a cut in G separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction. Hence, $v \notin V(B)$ and there exists $v_1 \in V(B(b_1, b'])$ such that v_1, v are incident with some finite face of G_0, and v, b_1' are incident with some finite face of G_0. But then, by (12), $\{v_1, v, b_1'\}$ is still a cut in G separating $\{a_0, b_1\}$ from $\{a_1, a_2, b_2\}$, a contradiction.

Now, by Lemma 2.2.1, we have $\alpha(A, B) = 2$ by the follwoing paths: the path $B_1 \cup e_8 \cup A[a_8, a_5] \cup e_5 \cup B[b_5, b_2''] \cup B_2$ from b_1 to b_2, the path $A[a_4, a_2] \cup e_4' \cup A_0$ from a_2 to a_0, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$ from a_1 to b_2. This is a contradiction. □

(14) (a) holds, $b_8 \neq b_4$, and G has no edge from $B[b_1, b_1')$ to $A(a_8, a_2]$.

First, (a) holds. For, otherwise, (d) holds by (8). So $b_1' \in B(b_4, b_5]$ and $b_2' \in B[b_7, b_2]$. By (1) and (5), $b_1^* \in B(b_5, b_2)$. Hence, $(f_1, e_6, e_5, e_4, e_8)$ (when $t_1 = b_4$) or $(f_1, e_6, e_5, f_4, e_8)$ (when $t_1 = a_4$) is a 5-edge configuration. However, by Lemma 2.1.9 and 2.5.3, G_0 has a cut contradicting (13) or the choice of $\{b_1', b_2'\}$.

Thus, $b_1' \in B[b_1, b_4]$. Since $b_8 \in B[b_1, b_1')$, $b_8 \neq b_4$. By (9), G has no edge from $B[b_1, b_1')$ to $A(a_5, a_2]$. Now, by the choice of e_8, G has no edge from $B[b_1, b_1')$ to $A(a_8, a_2]$. □

(15) G has no edge from $B(b_8, b_6)$ to $A[a_1, a_8)$, and so $(f_1, e_6, e_5, e_4, e_8)$ is a 5-edge con-

figuration with $b_1^* \in B[b_6, b_2)$.

First, suppose there exists $e = ab \in E(G)$ with $b \in V(B(b_8, b_6))$ and $a \in V(A[a_1, a_8))$. Then $a_7 \in A(a_1, a]$ to avoid the double cross e_4, e_7, e_8, e. But now, since $a_3 \in A[a_1, a_7)$, then $b_3 \in B(b_1, b_8]$ by (1), and so (e_3, e_8, e, e_6, e_7) contradicts the choice of $\mathcal{P}$.

Thus, $b_1^* \in B[b_6, b_2)$ and, hence, $(f_1, e_6, e_5, e_4, e_8)$ is a 5-edge configuration. $\square$

We choose f_1 so that $B[b_6, b_1^*]$ is minimal. Moreover, we let $e_5' = a_5'b_5' \in E(G)$ with $a_5' \in A(a_1^*, a_6)$ and $b_5' \in B[b_5, b_6)$ so that $B[b_5', b_6]$ is minimal. Now, since $(f_1, e_6, e_5', e_4, e_8)$ is a 5-edge configuration (by (15)), G_0 has a cut $S^\# := \{b_1^\#, b_2^\#\}$ or $S^\# := \{b_1^\#, b_2^\#, a_0^\#\}$ satisfying the conclusion of Lemma 2.5.3 (with respect to $(f_1, e_6, e_5', e_4, e_8)$), such that $b_1, b_1^\#, b_2^\#, b_2$ occur on B in order.

By (7), we have

(16) Conclusions (ii) and (iii) of Lemma 2.5.3 do not hold for $S^\#$ and $(f_1, e_6, e_5', e_4, e_8)$.

Case 1. (i) of Lemma 2.5.3 does not hold for $S^\#$ and $(f_1, e_6, e_5', e_4, e_8)$.

Then $b_1^\# \in B[b_1, b_8]$ and $b_2^\# \in B[b_5', b_6)$. By Lemma 2.1.9 and by the choice of $\{b_1', b_2'\}$, we have $b_1^\# = b_1, b_2' = b_2$, $a_0 = a_0'$, and $\alpha(A, B) \leq 1$. We further choose $\{b_1^\#, b_2^\#\}$ so that $B[b_2^\#, b_2]$ is minimal.

By the choice of $\{b_1', b_2'\}$ and the planar structure of G_0, $G_0 - a_0 - B(b_1, b_1')$ contains a path B_1 from b_1 to b_1'. Let $e_6' = a_6'b_6' \in E(G)$ with $a_6' \in A(a_5, a_2]$ and $b_6' \in B(b_2^\#, b_6]$, such that $A[b_6', b_2]$ is maximal.

Now G has no edge from $B(b_5', b_6')$ to A. For, suppose G has an edge from $B(b_5', b_6')$ to some $a \in V(A)$. Then $a \in A[a_1, a_5]$ by the choice of e_6', and $a \notin A(a_1^*, a_6)$ by the choice of e_5'. So $a \in A[a_1, a_1^*]$, contradicting (15).

Let A_0 be the path from a_0 to b_6' on the boundary of $G_0 - B[b_1, b_2^\#]$ without going through b_2. Since we are in Case 1, $A_0 \cap B(b_6, b_2] = \emptyset$ by the choice of $\{b_1^\#, b_2^\#\}$.

Note that there exists $e = ab \in E(G)$ with $a \in V(A[a_1, a_8))$ and $b \in V(B[b_1', b_2]) \setminus \{b_6\}$, such that e and e_7' are nonadjacent. For, otherwise, by (1) and (10), there exist $u \in$

$\{a_7', b_7'\}$ and a separation (G_1, G_2) in G, such that $V(G_1 \cap G_2) = \{b_1, b_1', a_8, b_6, u, a_1\}$, $A[a_1, a_8] \cup B[b_1, b_1'] \subseteq G_1$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)$, and $(G_1, b_1, b_1', a_8, b_6, u, a_1)$ is planar. This contradicts Lemma 2.1.3.

Then there exists $e_7'' = a_7''b_7'' \in E(G)$ with $a_7'' \in V(A(a_7', a_8))$ and $b_7'' \in V(B(b_6, b_7'))$. In fact, $b \notin B(b_8, b_6)$ by (15) and, hence, $b \in B(b_6, b_2]$. Thus, by (10) and the choice of e_7', $a \in A(a_7', a_8)$ and $b \in B(b_6, b_7')$. So e gives the desired e_7''.

We further choose e_7'' with $a_7'' \in A(a_7', a_8)$ and $b_7'' \in B(b_6, b_7')$ so that $A[a_1, a_7'']$ is maximal. Then $a_7'' \in A(a', a_8)$. For otherwise, $a_7'' \in A[a_1, a']$. By (10), (15), and the choice of e_7'', $\{b_1, b_1', a', a_8, b_6\}$ is a cut in G^* separating $A[a', a_8] \cup B[b_1, b_1']$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Note that $G_0 - A_0 - B[b_7', b_2)$ contains a path B_2 from b_2 to b_7''. For otherwise, $b_7' \neq b_2$, and there exist $v_1 \in V(A_0)$ and $v_2 \in V(B[b_7', b_2))$, such that v_1, v_2 are incident with some finite face in G_0. If $v_1 = a_0$ then $\{v_1, v_2, b_2\}$ is a cut in G^* separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction; if $v_1 \neq a$ then by (11), $\{b_1, b_2^{\#}, v_1, v_2, b_2\}$ is a cut in G^* separating a_0 from $\{a_1, a_2\}$, a contradiction.

Hence, $\alpha(A, B) = 2$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_5] \cup e_5 \cup A[a_7'', a_5] \cup e_7'' \cup B_2$ from b_1 to b_2, the path $A[a_6', a_2] \cup e_6' \cup A_0$ from a_2 to a_0, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$ from a_1 to b_2. This is a contradiction. □

Case 2. (i) of Lemma 2.5.3 holds for $S^{\#} := \{b_1^{\#}, b_2^{\#}, a_0^{\#}\}$ and $(f_1, e_6, e_5', e_4, e_8)$.

Then $b_1^{\#} \in B[b_1, b_8]$ and $b_2^{\#} \in B[b_6, b_1^*]$. Moreover, we choose $\{b_1^{\#}, b_2^{\#}\}$ so that $B[b_1^{\#}, b_2^{\#}]$ is maximal. By (7), G_0 contains a path from a_0 to $B(b_4, b_6)$ and internally disjoint from B. Then by Lemma 2.1.8 and the choice of $\{b_1', b_2'\}$, we have $b_1^{\#} = b_1, b_2' = b_2$, and one of the following holds:

(N1) $a_0 = a_0' = a_0^{\#}$, and so $c(A, B) \geq 2$.

(N2) $a_0^{\#} = a_0$, $b_2^{\#}$ is a cut vertex of G_0 separating b_2 from $\{a_0, b_1\}$, $a_0', a_0^{\#}, b_2^{\#}, b_2'$ are incident with some finite face of G_0; so $\alpha(A, B) \leq 1$.

(N3) $a_0' = a_0, b_1'$ is a cut vertex of G_0 separating b_1 from $\{a_0, b_2\}$, $a_0', a_0^{\#}, b_1^{\#}, b_1'$ are incident with some finite face of G_0; so $\alpha(A, B) \leq 1$.

In particular, there exists a vertex $a_0^* \in \{a_0', a_0^{\#}\}$, such that $\{b_1', b_2^{\#}, a_0^*\}$ is a 3-cut in G_0 separating $B[b_1', b_2^{\#}]$ from $\{a_0, b_1, b_2\}$. Let $e_9 = a_9b_9 \in E(G)$ with $b_9 \in B(b_1', b_2^{\#})$ and $a_9 \in A[a_1, a_2]$, such that $A[a_1, a_9]$ is minimal. There also exists $e_9' = a_9'b_9' \in E(G)$ with $a_9' \in V(A(a_9, a_2])$ and $b_9' \in V(B[b_1, b_1')) \cup V(B(b_2^{\#}, b_2])$; for otherwise, $\{a_0^*, b_1', b_2^{\#}, a_9, a_2\}$ is a cut in G separating $A[a_9, a_2] \cup B[b_1', b_2^{\#}]$ from $\{a_0, a_1, a_2, b_1, b_2\}$, a contradiction.

Note that $a_9 \notin A[a_1, a_8)$; for otherwise, $b_9 \notin B(b_8, b_6)$ by (15) and, hence, $b_9 \in B[b_6, b_2^{\#})$, contradicting the choice of f_1. Next, $b_9' \in B(b_2^{\#}, b_2]$; as otherwise, $a_9' \notin A(a_5, a_2]$ by (9) and, hence, $a_9' \in A(a_9, a_5]$, contradicting the choice of e_8. By (6), $a_9' \notin A(a_5, a_2]$; so $a_9' \in A(a_9, a_5]$. Furthermore, $b_9' \in B(b_2^{\#}, b_7]$; or else, $(e_3, e_4, e_5, e_6, e_9')$ contradicts the choice of $\mathcal{P}$.

Now, since $a_9' \in A(a_9, a_5]$, $a_9 \neq a_5$; so $a_9 \in A[a_8, a_5)$. Moreover, $b_9 \notin B(b_5, b_2^{\#})$ to avoid the double cross e_9', e_5, e_6, e_9. By (5), $b_9 \notin B(b_4, b_5]$. So $b_9 \in B(b_1', b_4]$.

We choose e_9' so that $B[b_2^{\#}, b_9']$ is minimal. Since $a_9' \in A(a_9, a_5]$, $a_5 \neq a_9$. Then we will derive a contradiction by showing that $\alpha(A, B) = 2$.

Subcase 2.1. (N1) holds.

By the choice of $\{b_1', b_2'\}$ and the planar structure of G_0, $G_0 - B(b_1, b_1') - a_0$ contains a path B_1 from b_1 to b_1'. Moreover, by the choice of $\{b_1^{\#}, b_2^{\#}\}$ and by planar structure of G_0, $G_0 - B(b_2^{\#}, b_2) - a_0$ contains a path B_2 from $b_2^{\#}$ to b_2.

Note that there exist $f_8 = a_8^*b_8^*, f_9 = a_9^*b_9^* \in E(G)$ with $a_8^*, a_9^* \in V(A(a_1, a_8))$ and $b_8^*, b_9^* \in V(B(b_1', b_2])$ such that $a_8^* \neq a_9^*$ and $b_8^* \neq b_9^*$. For otherwise, there exist $v \in V(G)$ and a separation (G_1, G_2) in G, such that $V(G_1 \cap G_2) = \{b_1', a_0, b_1, a_1, v, a_8\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $A(a_1, a_8) \cup B(b_1, b_1') \subseteq G_2$, and $(G_2, b_1', a_0, b_1, a_1, v, a_8)$ is planar. This contradicts Lemma 2.1.3.

Now $b_8^*, b_9^* \in B[b_6, b_2]$ by (15), and f_8, f_9 form a cross by (10). So a_1, a_8^*, a_9^*, a_2 occur on A in order, and b_1, b_9^*, b_8^*, b_2 occur on B in order. We further choose f_8, f_9 with $A[a_8^*, a_9^*]$ maximal. By the existence of e_9' and by (10), $b_8^* \in B(b_2^{\#}, b_2]$.

There exists $f_5 = a_5^* b_5^*$ with $b_5^* \in V(B[b_1, b_1'))$ and $a_5^* \in V(A(a_1, a_9^*))$. For otherwise, all edges from $B[b_1, b_1')$ will end in $\{a_1\} \cup V(A[a_9^*, a_8])$. By the choice of f_8, f_9, G has no edge from $A(a_9^*, a_8)$ to $B(b_8, b_2]$. Hence, G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{b_1', a_0, b_1, a_1, a_9^*, a_8\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $A(a_9^*, a_8) \cup B(b_1, b_1') \subseteq G_2$, and $(G_2, b_1', a_0, b_1, a_1, a_9^*, a_8)$ is planar. By Lemma 2.1.3, $|V(G_2 - G_1)| = 1$. So $V(G_2 - G_1) = \{b_8\}$, and G has edges from b_8 to $b_1', a_0, b_1, a_1, a_9^*, a_8$, respectively. But then, b_1 has degree 1 in G, a contradiction.

By (7), there exists a path A_0 from a_0 to $B(b_4, b_6)$ in G_0 and internally disjoint from B. Now, $\alpha(A, B) = 2$ and $c(A, B) = 0$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_9] \cup e_9 \cup A[a_9^*, a_9] \cup f_9 \cup B[b_9^*, b_2^{\#}] \cup B_2$ from b_1 to b_2, the path $B[b_1, b_5^*] \cup f_5 \cup A[a_5^*, a_8^*] \cup f_8 \cup B[b_8^*, b_2]$ from b_1 to b_2, and the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2. This is a contradiction.

Subcase 2.2. (N2) holds.

Then there exists $e_7'' = a_7'' b_7'' \in E(G)$ with $a_7'' \in V(A[a_1, a_8))$ and $b_7'' \in V(B(b_1', b_2])$ such that $a_7'' \neq a_7'$ and $b_7'' \neq b_7'$. For otherwise, by (1), (10) and (15), G has a separation (G_1, G_2), such that $V(G_1 \cap G_2) = \{v, a_8, b_1', a_0'\}$ with $v \in \{a_7', b_7'\}$, $a_0, a_1, b_1 \in V(G_2)$, $|V(G_2 - G_1)| \geq 4$, $a_2, b_2 \in V(G_1)$, and $(G_2, a_0, b_1, a_1, v, a_8, b_1', a_0')$ is planar. This contradicts Lemma 2.1.3 (when $v = a_7' = a_1$) or Lemma 2.1.4 (when $v \neq a_1$).

By (10) and (15), $a_7'' \in A(a_7', a_8)$ and $b_7'' \in B[b_6, b_7')$. We further choose e_7'' so that $A[a_1, a_7'']$ is maximal. Then $a_7'' \in A(a', a_8)$. For otherwise, $a_7'' \in A[a_1, a']$ and, by the choice of e_7'', G has no edge from $A(a', a_8)$ to $B(b_1', b_2]$. Hence, G has a separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a', a_8, b_1', a_0', a_0, b_1\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, a', a_8, b_1', a_0', a_0, b_1)$ is planar. This contradicts Lemma 2.1.3.

By the choice of $\{a_0^{\#}, b_1^{\#}, b_2^{\#}\}$ and the planar structure of G_0, $G_0 - B[b_7', b_2)$ contains a path B_2 from b_2 to $b_2^{\#}$. Let A_0 be the path from a_0 to $B(b_4, b_6)$ in G_0, which is internally disjoint from B. Moreover, we further choose A_0 such that $A_0[a_0, a_0']$ is on the boundary of G_0 without going through b_1.

Then $G_0 - B(b_1, b'] - A_0$ contains a path B_1 from b_1 to b_1'. For otherwise, $b_1' \neq b_1$ and there exist $v_1 \in V(A_0[a_0, a_0'])$ and $v_2 \in V(B(b_1, b'])$, such that v_1, v_2 are incident with some finite face of G_0. Now, by (12), $\{b_1, v_1, v_2, b_2\}$ (if $v_1 \neq a_0$) is a cut in G^* separating a_0 from $\{a_1, a_2\}$, or $\{v_1, v_2, b_1\}$ (if $v_1 = a_0$) is a cut in G^* separating $N_G(b_1)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. This is a contradiction.

Hence, $\alpha(A, B) = 2$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_9] \cup e_9 \cup A[a_7'', a_9] \cup e_7'' \cup B[b_7'', b_2^{\#}] \cup B_2$ from b_1 to b_2, the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a_7'] \cup e_7' \cup B[b_7', b_2]$. This is a contradiction.

Subcase 2.3. (N3) holds.

Then there exists $e_7'' = a_7''b_7'' \in E(G)$ with $a_7'' \in V(A(a', a_8))$ and $b_7'' \in V(B(b_1', b_2])$, such that $a_7'' \neq a_7'$ and $b_7'' \neq b_7'$. For otherwise, by (10) and (15), there exist $v \in \{a_7', b_7'\}$ and a separation (G_1, G_2) in G, such that $V(G_1 \cap G_2) = \{v, a', a_8, b_1, b_1'\}$, $A[a', a_8] \cup B[b_1, b_1'] \subseteq G_1$, and $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_2)$, a contradiction.

By (10) and (15), $a_7'' \in A(a_7', a_8)$ and $b_7'' \in B[b_6, b_7')$. By the choice of $\{a_0', b_1', b_2'\}$ and the planar structure of G_0, $G_0 - B(b_1, b']$ contains a path B_1 from b_1 to b_1'. Let A_0 be the path from a_0 to $B(b_4, b_6)$ in G_0, which is internally disjoint from B, and we choose A_0 such that $A_0[a_0, a_0^{\#}]$ is on the boundary of G_0 without going through b_2.

Then $G_0 - B[b_7', b_2) - A_0$ contains a path B_2 from b_2 to $b_2^{\#}$. For otherwise, $b_7' \neq b_2$, and there exist $v_1 \in V(A_0[a_0, a_0^{\#}])$ and $v_2 \in V(B[b_7', b_2))$, such that v_1, v_2 are incident with some finite face of G_0. Now, by (11), $\{b_1, v_1, v_2, b_2\}$ (if $v_1 \neq a_0$) is a cut in G^* separating a_0 from $\{a_1, a_2\}$, or $\{v_1, v_2, b_2\}$ (if $v_1 = a_0$) is a cut in G^* separating $N_G(b_2)$ from $\{a_0, a_1, a_2, b_1, b_2\}$. This is a contradiction.

Now, $\alpha(A, B) = 2$ by Lemma 2.2.1 and the following paths: the path $B_1 \cup B[b_1', b_9] \cup$

$e_9 \cup A[a''_7, a_9] \cup e''_7 \cup B[b''_7, b^{\#}_2] \cup B_2$ from b_1 to b_2, the path $A_0 \cup B(b_4, b_6) \cup e_5 \cup A[a_5, a_2]$ from a_0 to a_2, the path $A[a_1, a'] \cup e' \cup B[b_1, b']$ from a_1 to b_1, and the path $A[a_1, a'_7] \cup e'_7 \cup B[b'_7, b_2]$. This is a contradiction. $\square$

CHAPTER 3

FUTURE WORK

3.1 A characterization of two-three linked graphs

In fact, Robertson and Seymour asked for a characterization of two-three linked graphs. Here, we believe we have such a characterization, although it is quite complicated (even to state) and its proof is longer.

We say that $(G, a_0, a_1, a_2, b_1, b_2)$ is *reducible*, if one of the following holds:

(R1) G has an edge e with one end in $\{a_0, a_1, a_2\}$ and one end in $\{b_1, b_2\}$.

(R2) There exists a separation (G_1, G_2) in G of order at most 1.

(R3) There exists a separation (G_1, G_2) in G of order 2, satisfying one of the following properties:

  (a) $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$ and $V(G_2 - G_1) \neq \emptyset$; or

  (b) $|V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}| = 1$ and $|E(G_2)| \geq 3$; or

  (c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_i, b_j \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)$, and $(G_2, a_i, b_j, c_2, c_1)$ is planar; or

  (d) for some $j \in \{1, 2\}$ and some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_2, c_1)$ is planar; or

  (e) for some $i \in \{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2\}$, $a_i, b_1, b_2 \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_1, b_2\} \subseteq V(G_1)$, and $(G_2, b_1, a_i, b_2, c_2, c_1)$ is planar.

(R4) There exists a separation (G_1, G_2) in G of order 3, satisfying one of the following properties:

(a) $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$ and $V(G_2 - G_1) \neq \emptyset$; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $\{d\} = \{a_0, a_1, a_2, b_1, b_2\} \cap V(G_2 - G_1)$, (G_2, d, c_3, c_2, c_1) is planar, and $|V(G_2 - G_1)| \geq 2$; or

(c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $a_i, b_j \in V(G_2 - G_1)$, $\{a_0, a_1, a_2, b_1, b_2\} - \{a_i, b_j\} \subseteq V(G_1)$, $(G_2, a_i, b_j, c_1, c_2, c_3)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(d) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, and $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)$ is planar; or

(e) for some $i \in \{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3\}$, $b_1, a_i, b_2 \in V(G_2 - G_1)$, $\{a_0, a_1, a_2\} - \{a_i\} \subseteq V(G_1)$, and $(G_2, b_1, a_i, b_2, c_3, c_2, c_1)$ is planar.

(R5) There exists a separation (G_1, G_2) in G of order 4, satisfying one of the following properties:

(a) let W be a graph with $V(W) = \{w_0, w_1, w_2, w_3, w_4\}, E(W) = \{w_0 w_i; i = 1, 2, 3, 4\} \cup \{w_1 w_2, w_1 w_3\}$, then $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) \neq \emptyset$, and G_2 is not a subgraph of W; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, c_3, c_4, G has edges from c_1 to c_2, c_3, and for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $a_i, b_j \in V(G_1 \cap G_2)$; or

(c) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, a_i, b_j\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, a_i, b_j, and G has an edge from c_1 to c_2; or

(d) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, $V(G_2 - G_1) = \{c\}$, G has edges from c to c_1, c_2, c_3, c_4, G has an edge from c_1 to c_2, and for some permutation π of $\{0, 1, 2\}$, $\{a_{\pi(0)}, a_{\pi(1)}\} \subseteq V(G_1 \cap G_2)$ and $\{a_{\pi(0)}, a_{\pi(1)}\} \cap \{c_1, c_2\} \neq \emptyset$; or

(e) for some $i \in \{0, 1, 2\}$, $\{a_i\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{b_1, b_2, c_1, c_2\}$, $(G_2, a_i, b_1, c_1, c_2, b_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(f) for some permutation π of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{b_j\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{a_{\pi(1)}, a_{\pi(2)}, c_1, c_2\}$, $(G_2, b_j, a_{\pi(1)}, c_1, c_2, a_{\pi(2)})$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(g) for some permutation π of $\{0, 1, 2\}$ and some $j \in \{1, 2\}$, $\{a_{\pi(0)}\} = V(G_2 - G_1) \cap \{a_0, a_1, a_2, b_1, b_2\}$, $V(G_1 \cap G_2) = \{b_j, a_{\pi(1)}, c_1, c_2\}$, $(G_2, a_{\pi(0)}, b_j, c_1, a_{\pi(1)}, c_2)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(h) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, c_1, c_2, a_{\pi(0)}, c_3, a_{\pi(1)}, b_j)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(i) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(0)}\}$, $a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(j) for some $i \in \{0, 1, 2\}$ and some $j \in \{1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, b_j\}$, $a_i, b_{3-j} \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_i \subseteq V(G_1)$, $(G_2, b_{3-j}, a_i, b_j, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 3$; or

(k) for some permutation π of $\{0, 1, 2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_{\pi(0)}, a_{\pi(1)}, b_j \in V(G_2 - G_1)$, $a_{\pi(2)}, b_{3-j} \in V(G_1)$, $(G_2, a_{\pi(0)}, b_j, a_{\pi(1)}, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(l) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_i, b_1, b_2 \in V(G_2 - G_1)$, $\{a_1, a_2, a_3\} - a_i \subseteq V(G_1)$, $(G_2, b_1, a_i, b_2, c_4, c_3, c_2, c_1)$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(m) for some permutation π of $\{0,1,2\}$, $a_{\pi(0)}, a_{\pi(1)}, b_1, b_2 \in V(G_1)$, $\{a_{\pi(0)}, a_{\pi(1)}, b_1, b_2\} \cap V(G_2) \neq \emptyset$, $a_{\pi(2)} \in V(G_2) - V(G_1)$, and G_1 has a disk representation in which $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2$ occur on the boundary of the disk in the order listed and the vertices in $V(G_1) \cap V(G_2)$ are incident with a common finite face.

(R6) There exists a separation (G_1, G_2) in G of order 5, satisfying one of the following properties:

(a) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, $E(G[\{c_1, c_2, c_3, c_4, c_5\}]) \subseteq E(G_1)$, $(G_2, c_1, c_2, c_3, c_4, c_5)$ is planar, and $|V(G_2 - G_1)| \geq 2$; or

(b) $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4, c_5\}$, $\{a_0, a_1, a_2, b_1, b_2\} \subseteq V(G_1)$, and for some permutation π of $\{0,1,2\}$, G_1 has a disk representation with the vertices $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, c_1, c_2, c_3, c_4, c_5$ drawn on the boundary of the disk in the order listed; or

(c) for some permutation π of $\{0,1,2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, b_1, b_2, a_{\pi(1)}\}$, $a_{\pi(2)} \in V(G_1 - G_2)$, $a_{\pi(0)} \in V(G_2 - G_1)$, $(G_2, b_1, c_1, a_{\pi(1)}, c_2, b_2, a_{\pi(0)})$ is planar, and $|V(G_2 - G_1)| \geq 4$; or

(d) for some $j \in \{1,2\}$ and some permutation π of $\{0,1,2\}$, $V(G_1 \cap G_2) = \{c_1, c_2, c_3, a_{\pi(1)}, b_j\}$, $a_{\pi(2)} \in V(G_1 - G_2)$, $a_{\pi(0)}, b_{3-j} \in V(G_2 - G_1)$, $(G_2, a_{\pi(1)}, c_1, c_2, c_3, b_j, a_{\pi(0)}, b_{3-j})$ is planar, and $|V(G_2 - G_1)| \geq 3$.

Actually, we can prove that if $(G, a_0, a_1, a_2, b_1, b_2)$ is reducible, then we could either easily determine whether or not $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible, or reduce $(G, a_0, a_1, a_2, b_1, b_2)$ to $(G', a'_0, a'_1, a'_2, b'_1, b'_2)$ with $(|V(G)|, |E(G)|) > (|V(G')|, |E(G')|)$ in lexicographic order, such that $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible iff $(G', a'_0, a'_1, a'_2, b'_1, b'_2)$ is feasible.

With all these, we can state our main result.

Theorem 3.1.1 *Let $(G, a_0, a_1, a_2, b_1, b_2)$ be a rooted graph. Then one of the following conclusions holds:*

(C1) There exists a cluster $\{X_1, X_2\}$ *in* G *such that* $\{a_0, a_1, a_2\} \subseteq X_1$ *and* $\{b_1, b_2\} \subseteq X_2$.

(C2) $(G, a_0, a_1, a_2, b_1, b_2)$ *is reducible.*

(C3) For some $i \in \{0, 1, 2\}$, $G - a_i$ *has no cluster* $\{X_1, X_2\}$ *such that* $\{a_0, a_1, a_2\} - \{a_i\} \subseteq X_1$ *and* $\{b_1, b_2\} \subseteq X_2$.

(C4) There exist a permutation π *of* $\{0, 1, 2\}$, *a graph* H *and vertices* $s, t, s', t' \in V(H)$ *such that* G *is obtained from* H *by identifying* s *with* s' *and* t *with* t', *respectively, and* H *has a disk representation with the vertices* $a_{\pi(0)}, b_1, a_{\pi(1)}, b_2, a_{\pi(2)}, s, t, s', t'$ *drawn on the boundary of the disk in the order listed.*

(C5) G *has a separation* (G_1, G_2) *in* G *of order 4, such that* $V(G_1 \cap G_2) = \{c_1, c_2, c_3, c_4\}$, $a_0, a_1, a_2, b_1, b_2 \in V(G_1)$, *and there exist a permutation* π *of* $\{0, 1, 2\}$, *a graph* H *and vertices* $c_2', c_2'' \in V(H)$, *where* G_1 *is obtained from* H *by identifying* c_2' *with* c_2'', $(H, a_{\pi(1)}, b_1, a_{\pi(0)}, b_2, a_{\pi(2)}, c_2'', c_4, c_3, c_2', c_1)$ *is planar, and* $c_2 \in V(G_1)$ *is the vertex obtained by identifying* c_2' *with* c_2''.

3.2 Clarifying (C3)

Note that if (C4) or (C5) holds, then (C1) will not hold. However, if (C3) holds, $(G, a_0, a_1, a_2, b_1, b_2)$ may be feasible or may be infeasible. Although by using 2-linkage algorithms, it is easy to judge whether $(G, a_0, a_1, a_2, b_1, b_2)$ admits (C3), we want to give a more precise characterization of feasible rooted graphs when (C3) holds.

We will still assume G is not reducible. So by applying Seymour's version of 2-linkage theorem in [37], when (C3) holds, there exists $i \in \{0, 1, 2\}$, such that $(G - a_i, a_{i+1}, b_1, a_{i-1}, b_2)$ is planar. So G actually is an apex graph.

3.3 A practical algorithm

Another possible future work is to develop a practical polynomial time algorithm for the two-three linkage problem.

Note that the existence of such an algorithm with polynomial running time is guaranteed by the work of Robertson and Seymour in [40]: Given a graph G and $k \geq 1$ pairs of vertices $\{s_i, t_i\}$, $i = 1, \cdots, k$ of G with k fixed, there exists a polynomial time algorithm for deciding if there are k mutually internally vertex-disjoint paths in G joining s_i and t_i, $i = 1, \cdots, k$. In fact, to resolve the two-three linkage problem, we just need to check:

(i) whether for some $i \in \{0, 1, 2\}$, G contains 3 mutually internally vertex-disjoint paths joining the pairs $\{b_1, b_2\}$, $\{a_{i-1}, a_i\}$ and $\{a_i, a_{i+1}\}$; or

(ii) whether for some vertex $v \in V(G) - \{a_0, a_1, a_2, b_1, b_2\}$, G contains 4 mutually vertex-disjoint paths to join the pairs $\{b_1, b_2\}$, $\{v, a_0\}$, $\{v, a_1\}$ and $\{v, a_2\}$.

Clearly, the answer is yes iff $(G, a_0, a_1, a_2, b_1, b_2)$ is feasible. The disjoint paths algorithm of Robertson and Seymour has running time $O(|V(G)|^3)$. So the above algorithm runs $O(|V(G)|^4)$ time.

However, the disjoint paths algorithm of Robertson and Seymour is not practical, since it involves an enormous constant. Hence, it is meaningful to come up with a practical algorithm for the two-three linkage problem. In fact, to the best of our knowledge, Tholey [41] found the $O(|E(G)| + |V(G)|\alpha(|V(G)|, |V(G)|))$-time algorithm, the currently best known nearly linear time bound, of 2-linkage problem, where α denotes the inverse of the Ackermann function. By repeatedly using 2-linkage algorithm, we expect to obtain a $O(|V(G)|^3)$-time two-three linkage algorithm.

3.4 A related conjecture

A graph G is apex if $G - v$ is planar for some vertex $v \in V(G)$. Jørgensen [34] conjectured that every 6-connected graph with no K_6-minor is apex.

In the two-three linkage problem, we only consider finding disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$ and $\{b_1, b_2\} \subseteq V(G_2)$. However, it is also natural to ask whether we can find such disjoint connected subgraphs G_1, G_2 satisfying additional properties. For example, we have the following conjecture.

Conjecture 3.4.1 *Any 6-connected non-apex graph G with distinct vertices $a_0, a_1, a_2, b_1, b_2 \in V(G)$ contains disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and the following properties hold:*

(P1) there exists a vertex $v \in V(G_1) - \{a_0, a_1, a_2\}$ such that G_1 has three internally disjoint paths from v to a_0, a_1, a_2, respectively;

(P2) for each vertex $v \in G_1$, $\{a_0, a_1, a_2\} - \{v\}$ are contained in one component of $G_1 - v$.

One observation is that if $(G - a_0, a_1, b_1, a_2, b_2)$ is planar, then there do not exist disjoint connected subgraphs G_1, G_2 in G such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and G_1 satisfies (P1) and (P2). Note that such G is apex, and G can be 6-connected.

If Conjecture 3.4.1 is true, we may prove that given a 6-connected graph G and triangles $a_i b_1 b_2 a_i$ for $i = 0, 1, 2$, $G - b_1 b_2 - \{a_i b_j : i = 0, 1, 2$ and $j = 1, 2\}$ contains disjoint connected subgraphs G_1, G_2 such that $\{a_0, a_1, a_2\} \subseteq V(G_1)$, $\{b_1, b_2\} \subseteq V(G_2)$, and G_1 satisfies (P1) and (P2). Such properties could be useful in resolving Jørgensen's conjecture for 6-connected graph in which some edge is contained in three triangles.

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